2.7 The Rayleigh-Ritz formulation for the problem of solving Eq. (2.1) with boundary conditions (2.2) and (2.3) can be stated as: Minimize the functional

$$F(T)=\int_0^L\left\{\frac{K}{2}\left[\frac{dT}{dx}\right]^2-QT\right\}dx-Tq\,\big|_{x=0}$$

over all functions $T(x)$ with square integrable first derivatives that satisfy Eq. (2.3) at $x = L$. Approximate $T(x)$ using two linear elements as we did before and replace into Eq. (2.1) to obtain $F(T) \cong F(a_1, a_2, a_3)$. Now minimize $F(a_1, a_2, a_3)$ as a function of three variables. Show that the final system of equations is identical to Eq. (2.31), thus, in this case, the Galerkin method and the Rayleigh-Ritz method are equivalent.

Solution:

Element1:

$$\phi_1=1-\frac{2}{L}x \qquad \phi_2=\frac{2}{L}x$$

$$\int_0^{L/2}\frac{K}{2}\left(-\frac{2}{L}a_1+\frac{2}{L}a_2\right)^2-Q\left[\left(1-\frac{2}{L}x\right)a_1+\frac{2}{L}xa_2\right]dx-a_1q=\frac{K}{L}\left(a_2-a_1\right)^2-\frac{QL}{4}\left(a_1+a_2\right)-a_1q$$

(1)

Element2:

$$\phi_2=2-\frac{2}{L}x \qquad \phi_3=\frac{2}{L}x-1$$

$$\int_{L/2}^{L}\frac{K}{2}\left(-\frac{2}{L}a_2+\frac{2}{L}a_3\right)^2-Q\left[\left(2-\frac{2}{L}x\right)a_2+\left(\frac{2}{L}x-1\right)a_3\right]dx-a_1q=\frac{K}{L}\left(a_3-a_2\right)^2-\frac{QL}{4}\left(a_2+a_3\right)$$

(2)

$$F(T)=(1)+(2)=\frac{K}{L}\left(a_2-a_1\right)^2+\frac{K}{L}\left(a_3-a_2\right)^2-\frac{QL}{4}\left(a_1+2a_2+a_3\right)-a_1q$$

$$F(T)=F(a_1,a_2,a_3)=\frac{K}{L}\left(a_1^2-2a_1a_2+2a_2^2-2a_2a_3+a_3^2\right)$$

The conditions for a minimum are $\dfrac{\partial F}{\partial a_1}=\dfrac{\partial F}{\partial a_2}=\dfrac{\partial F}{\partial a_3}=0$ thus

$$\frac{\partial F}{\partial a_1}=\frac{2K}{L}\left(a_1-a_2\right)-\frac{QL}{4}-q=0$$

$$\frac{\partial F}{\partial a_2}=\frac{2K}{L}\left(-a_1+2a_2-a_2\right)-\frac{QL}{2}=0$$

$$\frac{\partial F}{\partial a_3}=\frac{2K}{L}\left(a_3-a_2\right)-\frac{QL}{4}=0$$

In Matrix form $\dfrac{2K}{L}\begin{bmatrix}1&-1&0\\-1&2&-1\\0&-1&1\end{bmatrix}\begin{Bmatrix}a_1\\a_2\\a_3\end{Bmatrix}=\dfrac{QL}{4}\begin{Bmatrix}1\\2\\1\end{Bmatrix}+\begin{Bmatrix}q\\0\\0\end{Bmatrix}$ which is identical to Eq. 2.31.