2.5 Use the weak statement formulation to find solutions to the equation

$$\frac{d^2u}{dx^2} = 1, \qquad 0 < x < 1$$

$$u(0) = 0, \qquad \left.\frac{du}{dx}\right|_{x=1} = 1$$

(a) Using linear interpolation functions, as explained in Section 2.3, use (i) one element and (ii) two elements.

(b) Using simple polynomial functions that satisfy the boundary conditions at the left-hand side, i.e.,

(i) $u(x) = a_1 x$, hence $\phi_1(x) = W_1(x) = x$

(ii) $u(x) = a_1 x + a_2 x^2$, hence $\phi_1(x) = W_1(x) = x$,

$\phi_2(x) = W_2(x) = x^2$.

(c) Using circular functions that satisfy the boundary conditions at the left-hand side, i.e.,

(i) $u(x) = a_l \sin(\pi/2Lx)$

(ii) $u(x) = a_l \sin(\pi/2Lx) + aa_2 \sin(3\pi/2Lx)$

(d) Find the analytical solution and compare with results from (a), (b) and (c).

Solution

a) The residual function is $R(u,x) = \dfrac{d^2u}{dx^2} + 1$ The weak form after integration by parts is

$$\int_0^1 \left(\frac{dW}{dx}\frac{du}{dx} + W \right) dx - W\left.\frac{du}{dx}\right|_0^1 = 0$$

i)

One element solution

$$\phi_1 = 1 - x \qquad \phi_2 = x$$

$$u(x) = (1-x)a_1 + x a_2$$

$$\int_0^1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix} dx \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} + \int_0^1 \begin{bmatrix} 1-x \\ x \end{bmatrix} dx - \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Integrating

$$a_1 - a_2 = -1/2$$
$$-a_1 + a_2 = 1/2$$

since $a_1 = 0$ then $a_2 = 1/2$ *or* $u(x) = x/2$

ii)

Two element solution. Element 1:

$$\phi_1 = 1-2x \qquad \phi_2 = 2x$$

$$\int_0^{1/2}\begin{bmatrix}-2\\2\end{bmatrix}\begin{bmatrix}-2 & 2\end{bmatrix}dx\begin{Bmatrix}a_1\\a_2\end{Bmatrix}+\int_0^{1/2}\begin{bmatrix}1-2x\\2x\end{bmatrix}dx-\begin{Bmatrix}0\\0\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix} \quad or \quad \begin{bmatrix}2 & -2\\-2 & 2\end{bmatrix}\begin{Bmatrix}a_1\\a_2\end{Bmatrix}=\begin{Bmatrix}-1/4\\-1/4\end{Bmatrix}$$

Element 2

$$\phi_2 = 2(1-x) \qquad \phi_3 = 2(x-1/2)$$

$$\int_{1/2}^{1}\begin{bmatrix}-2\\2\end{bmatrix}\begin{bmatrix}-2 & 2\end{bmatrix}dx\begin{Bmatrix}a_2\\a_3\end{Bmatrix}+\int_{1/2}^{1}\begin{bmatrix}2(1-x)\\2(x-1/2)\end{bmatrix}dx-\begin{Bmatrix}0\\1\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix} \quad or \quad \begin{bmatrix}2 & -2\\-2 & 2\end{bmatrix}\begin{Bmatrix}a_2\\a_3\end{Bmatrix}=\begin{Bmatrix}-1/4\\3/4\end{Bmatrix}$$

Assemble the elements

$$\begin{bmatrix}2 & -2 & 0\\-2 & 4 & -2\\0 & -2 & 2\end{bmatrix}\begin{Bmatrix}a_1\\a_2\\a_3\end{Bmatrix}=\begin{Bmatrix}-1/4\\-1/2\\3/4\end{Bmatrix} \quad \text{since} \quad a_1=0 \;\; we\,get \;\; a_2=1/8 \;\; and \;\; a_3=1/2$$

$$u(x)=\begin{cases} x/4 & 0\le x\le 1/2\\ \dfrac{3x-1}{4} & 1/2\le x\le 1\end{cases}$$

b)

Using polynomials

i)

$$\phi_1 = x \qquad u(x)=a_1x$$

$\int_0^1(1\cdot a_1+x)\,dx-1=0$, $a_1=1/2$, $u(x)=x/2$, same as using one element.

ii)

$\phi_1 = x$, $\phi_2 = x^2$, $u(x)=a_1x+a_2x^2$

For W=ϕ_1 $\int_0^1\left[1\cdot(a_1+2a_2x)+x\right]dx-1=0$

For W=ϕ_2 $\int_0^1\left[2x\cdot(a_1+2a_2x)+x^2\right]dx-1=0$

The system becomes

$$a_1+a_2=1/2$$

$$a_1+\frac{4}{3}a_2=2/3 \qquad or \qquad a_1=0 \;,\; a_2=1/2 \;\; and \;\; u(x)=x^2/2$$

c)

Using circular functions

i)

$$\phi_1 = \sin\left(\frac{\pi}{2}x\right) \quad , \quad u(x) = a_1 \sin\left(\frac{\pi}{2}x\right) \quad , \quad \phi_1' = \frac{\pi}{2}\cos\left(\frac{\pi}{2}x\right)$$

$$\int_0^1 \left[\frac{\pi^2}{4} a_1 \cos^2\left(\frac{\pi}{2}x\right) + \sin\left(\frac{\pi}{2}x\right)\right] dx - \sin\left(\frac{\pi}{2}x\right)\frac{du}{dx}\bigg|_0^1 = 0$$

Use the identities $\int \cos^2 \alpha x \, dx = \frac{1}{2\alpha}(\sin\alpha x \, \cos\alpha x - \alpha x)$ and $\int \sin\left(\frac{\pi}{2}x\right) dx = -\frac{2}{\pi}\cos\left(\frac{\pi}{2}x\right)$ .Then

$$\frac{\pi^2 a_1}{4}\left(\sin\left(\frac{\pi}{2}x\right)\cos\left(\frac{\pi}{2}x\right) - \frac{x}{2}\right)\bigg|_0^1 + \left(-\frac{2}{\pi}\cos\left(\frac{\pi}{2}x\right)\right)\bigg|_0^1 - \sin\left(\frac{\pi}{2}x\right)\frac{du}{dx}\bigg|_0^1 = 0 \quad or$$

$$\frac{\pi^2 a_1}{8} + \frac{2}{\pi} - 1 = 0 \qquad \text{hence} \quad a_1 \approx 0.295 \quad \text{and} \quad u(x) = 0.295\sin\left(\frac{\pi}{2}x\right)$$

ii)

$$\phi_2 = \sin\left(\frac{3\pi}{2}x\right) \quad , \quad u(x) = a_1 \sin\left(\frac{\pi}{2}x\right) + a_2 \sin\left(\frac{3\pi}{2}x\right).$$

For $W = \phi_1$:
$$\int_0^1 \left[\frac{\pi^2}{4}\cos\left(\frac{\pi}{2}x\right)\left(a_1\cos\left(\frac{\pi}{2}x\right) + a_2\cos\left(\frac{3\pi}{2}x\right)\right) + \sin\left(\frac{\pi}{2}x\right)\right] dx - \sin\left(\frac{\pi}{2}x\right)\frac{du}{dx}\bigg|_0^1 = 0$$

For $W = \phi_2$:
$$\int_0^1 \left[3\frac{\pi^2}{4}\cos\left(3\frac{\pi}{2}x\right)\left(a_1\cos\left(\frac{\pi}{2}x\right) + a_2\cos\left(\frac{3\pi}{2}x\right)\right) + \sin\left(\frac{3\pi}{2}x\right)\right] dx - \sin\left(\frac{3\pi}{2}x\right)\frac{du}{dx}\bigg|_0^1 =$$

$$\frac{\pi^2}{8}a_1 + 0 \cdot a_2 + \frac{2}{\pi} - 1 = 0 \quad (a_1 \text{ is unchanged})$$

$$0 \cdot a_1 + \frac{9\pi^2}{8}a_2 + \frac{2}{3\pi} + 1 = 0 \quad , \quad a_2 \approx -0.109 \quad and$$

$$u(x) \approx 0.295\sin\left(\frac{\pi}{2}x\right) - 0.109\sin\left(\frac{3\pi}{2}x\right)$$

d) Analytical solution

$\frac{d^2u}{dx^2} = 1$,integrating $u(x) = \frac{x^2}{2} + c_1 x + c_2$ and applying boundary conditions $u^*(x) = \frac{x^2}{2}$.

Notice that all the approximations are exact at x = 1/2, and the Galerkin approximations using x and x^2 both yield the exact solution. Also notice that the boundary condition at x = 1 is satisfied only approximately. In case a), the first solution gives $\left.\frac{du}{dx}\right|_{x=1} = \frac{1}{2}$ and the second solution gives $\left.\frac{du}{dx}\right|_{x=1} = \frac{3}{4}$ which is getting better.This point is discussed in more detail in problem 2.6.