2.9 In example 2.2, first find the exact analytical solution, then calculate two more Galerkin approximations using

Solution:

The analytical solution is $u(x)=\frac{1}{2}\left(x-x^2\right)\cdot$ The Galerkin approximation is

$$\int_0^1\left(\frac{dW}{dx}\frac{du}{dx}-W\right)dx=0\cdot$$

$$u(x)=a_1\sin\pi x+a_2\sin 2\pi x\ ,\ W_1=\sin\pi x\ ,\ W_2=\sin 2\pi x$$

a)
$$W_1:\ \int_0^1\left[\pi^2\cos\pi x\left(a_1\cos\pi x+2a_2\cos 2\pi x\right)-\sin\pi x\right]dx=0$$

$$W_2:\ \int_0^1\left[2\pi^2\cos 2\pi x\left(a_1\cos\pi x+2a_2\cos 2\pi x\right)-\sin 2\pi x\right]dx=0 \quad or$$

$$\frac{\pi^2}{2}a_1+0\cdot a_2-\frac{2}{\pi}=0\ ,\ a_1=\frac{4}{\pi^3}\quad ,\ and\quad 0\cdot a_1+2\pi^2a_2+0=0\ ,\ a_2=0$$

$$u(x)=\frac{4}{\pi^3}\sin\pi x$$

b) $u(x)=a_1\sin\pi x+a_2\sin 2\pi x+a_3\sin 3\pi x\ ,\ W_1=\sin\pi x\ ,\ W_2=\sin 2\pi x\ ,\ W_3=\sin 3\pi x$

We only need to calculate a_3 , a_1 *and* a_2 remain the same due to the orthogonality of the sine functions.

$$W_3:\ \int_0^1\left[3\pi^2\cos 3\pi x\left(a_1\cos\pi x+2a_2\cos 2\pi x+3a_3\cos 3\pi x\right)-\sin 3\pi x\right]dx=0$$

$$\frac{9\pi^2}{2}a_3-\frac{2}{3\pi}=0\ ,\ a_3=\frac{4}{27\pi^3}\ ,\ u_3(x)=\frac{4}{3\pi^3}\left(\sin\pi x+\frac{1}{27}\sin 3\pi x\right)$$