2.6 Subdivide the interval $0 \le x \le L$ into 10 linear elements, construct the element equations for four consecutive elements starting with element four, and show that none of the parameters a_i is related to more than two additional parameters in the equations. The only ones involved being a_{i-1} and a_{i+1} (as a consequence of this, the final system of linear equations will involve a tri-diagonal matrix which is easy to solve).

Solution:

For an element $e_k = \{x / x_k \le x \le x_{k+1})$ the element stiffness matrix is obtained from $\int_{x_k}^{x_{k+1}} \frac{d\phi_i}{dx}\frac{d\phi_j}{dx}dx$. All the stiffness matrices must be equal, so let's calculate the first one.

$$\phi_1 = 1 - \frac{10}{L}x \qquad \phi_2 = \frac{10}{L}x \quad thus \int_0^{L/10} K \begin{bmatrix} -\frac{10}{L} \\ \frac{10}{L} \end{bmatrix} \begin{bmatrix} -\frac{10}{L} & \frac{10}{L} \end{bmatrix} dx = \frac{10K}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

and after assembling
$$\frac{10K}{L} \begin{bmatrix} 1 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \\ a_9 \\ a_{10} \\ a_{11} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Note that each equation i involves only three unknowns $a_{i-1}, a_i \; and \; a_{i+1}$, so the stiffness matrix is tri-diagonal.