2.4 Consider the equation

$$\frac{d^2u}{dx^2}+u+x=0, \qquad 0<x<1$$

with $u(0) = u(1) = 0$. Assume an approximation for $u(x) = a_1\phi_1(x)$ with $\phi_1(x) = x(1 - x)$. The residual is

$$R(u,x)=-2a_1+a_1[x(1-x)]+x$$

and the weight $W_1(x) = \phi_1(x) = x(1 - x)$. Find the solution from the following integral

$$\int_0^1 W(x)R(u,x)dx=0$$

.Solution:

$$\int_0^1 W(x)R(u,x)dx=\int_0^1 x(1-x)(-2a_1+a_1x(1-x)+x)dx=0$$

Or

$$\int_0^1 W(x)R(u,x)dx=\int_0^1(-2a_1x+(3a_1+1)x^2-2a_1+1)x^3+a_1x^4)dx=0$$

$$-a_1+\frac{1}{3}(3a_1+1)-\frac{1}{4}(2a_1+1)+\frac{1}{5}a_1=-\frac{3}{10}a_1+\frac{1}{12}=0$$

$$a_1=\frac{5}{18} \quad \text{therefore} \quad u(x)\approx\frac{5}{18}(x-x^2)$$

Compare with the exact solution given by

$$u^*(x)=\frac{\sin x}{\sin 1}-x$$

Evaluating at x=1/2 $u(1/2)=0.069444$ less than 5% error.

$$u^*(1/2)=0.069747$$