Chapter Two

EQUATIONS FOR STEADY ONE-DIMENSIONAL COMPRESSIBLE FLUID FLOW

SUMMARY OF MAJOR EQUATIONS

Relation Between Fractional Changes in Flow Variables

Mass Conservation:

$$\frac{d\rho}{\rho} + \frac{dV}{V} + \frac{dA}{A} = 0 \tag{2.3}$$

Momentum:

$$-\frac{dp}{\rho} = V\,dV \tag{2.8}$$

Energy:

$$c_p\,dT + V\,dV = 0 \tag{2.17}$$

Equation of State:

$$\frac{dp}{p} - \frac{d\rho}{\rho} - \frac{dT}{T} = 0 \tag{2.19}$$

Entropy:

$$\frac{ds}{c_p} = \frac{dT}{T} - \left(\frac{\gamma - 1}{\gamma}\right)\frac{dp}{p} \tag{2.24}$$

PROBLEM 2.1

Air enters a tank at a velocity of 100 m/s and leaves the tank at a velocity of 200 m/s. If the flow is adiabatic, find the difference between the temperature of the air at the exit and the temperature of the air at the inlet.

SOLUTION

Because the flow is adiabatic, the energy equation gives:

$$\left(c_p T_{exit} + \frac{V_{exit}^2}{2}\right) = \left(c_p T_{inlet} + \frac{V_{inlet}^2}{2}\right)$$

Hence:

$$T_{exit} - T_{inlet} = \left(\frac{1}{c_p}\right)\left(\frac{V_{inlet}^2}{2} - \frac{V_{exit}^2}{2}\right)$$

Since air flow is being considered the specific heat, c_p , will be assumed to be 1007 J/kg ^{o}C. The above equation then gives:

$$T_{exit} - T_{inlet} = \left(\frac{1}{1007}\right)\left(\frac{100^2}{2} - \frac{200^2}{2}\right) = -14.9^{o}\mathrm{C}$$

Therefore the temperature decreases by 14.9^{o}C.

PROBLEM 2.2

Air at a temperature of 25° C is flowing at a velocity of 500 m/s. A shock wave (see later chapters) occurs in the flow reducing the velocity to 300 m/s. Assuming the flow through the shock wave to be adiabatic, find the temperature of the air behind the shock wave.

SOLUTION

Because the flow is adiabatic, the energy equation gives if subscripts 1 and 2 refer to conditions before and after the shock wave respectively:

$$c_p T_2 + \frac{V_2^2}{2} = c_p T_1 + \frac{V_1^2}{2}$$

Hence:

$$T_2 = T_1 + \left(\frac{1}{c_p}\right)\left(\frac{V_1^2}{2} - \frac{V_2^2}{2}\right) = 298 + \left(\frac{1}{1007}\right)\left(\frac{500^2}{2} - \frac{300^2}{2}\right) = 377\text{K} = 104.4^{\text{o}}\,\text{C}$$

Since air flow is being considered the specific heat c_p has been assumed to be 1007 J/ kg $^{\text{o}}$C.

Therefore the temperature "behind" the shock wave is 104.4$^{\text{o}}$ C.

PROBLEM 2.3

Air being released from a tire through the valve is found to have a temperature of 15°C. Assuming that the air in the tire is at the ambient temperature of 30°C find the velocity of the air at the exit of the valve. The process can be assumed to be adiabatic.

SOLUTION

Because the flow is adiabatic, the energy equation gives if subscripts 1 and 2 refer to conditions in the tire and at the discharge from the valve respectively:

$$c_p T_2 + \frac{V_2^2}{2} = c_p T_1 + \frac{V_1^2}{2}$$

But the velocity in the tire can be assumed to be zero so this gives:

$$c_p T_2 + \frac{V_2^2}{2} = c_p T_1 \ , \quad \text{i.e.,} \ \frac{V_2^2}{2} = c_p\left(T_1 - T_2\right)$$

Hence, assuming that for air the specific heat c_p is 1007 J/ kg ^{o}C:

$$V_2 = \sqrt{2c_p\left(T_1 - T_2\right)} = \sqrt{2 \times 1007 \times (303 - 288)} = 173.8 \text{ m/s}$$

Therefore the air leaves the valve with a velocity of 173.8 m/s.

PROBLEM 2.4

A gas with a molecular weight of 4 and a specific heat ratio of 1.67 flows through a variable area duct. At some point in the flow the velocity is 180 m/s and the temperature is 10° C. At some other point in the flow, the temperature is -10° C. Find the velocity at this point in the flow assuming that the flow is adiabatic.

SOLUTION

Because the flow is adiabatic, the energy equation gives if subscripts 1 and 2 refer to conditions at the first and second points considered respectively:

$$c_p T_2 + \frac{V_2^2}{2} = c_p T_1 + \frac{V_1^2}{2}$$

Hence:

$$V_2^2 = V_1^2 + 2c_p\left(T_1 - T_2\right)$$

Assuming that the gas can be treated as a perfect gas:

$$R = c_p - c_v = c_p\left(1 - \frac{c_v}{c_p}\right) \quad \text{i.e.,} \quad c_p = \frac{\gamma R}{\gamma - 1}$$

Hence for the gas being considered:

$$c_p = \frac{1.67 \times (8314/4)}{1.67 - 1} = 5181 \text{ J/kg}^{\circ}\text{C}$$

The energy equation therefore gives:

$$V_2^2 = V_1^2 + 2c_p\left(T_1 - T_2\right) = 180^2 + 2 \times 6181 \times \left(283 - 263\right)$$

Which gives V_2 = 489.5 m/s. Therefore the velocity at the second point is 489.5 m/s.

PROBLEM 2.5

At a section of a circular duct through which air is flowing the pressure is 150 kPa , the temperature is 35 °C , the velocity is 250 m/s , and the diameter is 0.2 m. If, at this section, the duct diameter is increasing at a rate of 0.1 m / m find *dp/dx* , *dV/dx* , and *dρ/dx* .

SOLUTION

Because:

$$A = \frac{\pi D^2}{4}$$

it follows that:

$$\frac{1}{A}\frac{dA}{dx} = \frac{2}{D}\frac{dD}{dx}$$

Hence. at the section considered:

$$\frac{1}{A}\frac{dA}{dx} = \frac{2}{0.2} \times 0.1 = 1\text{m}^{-1}$$

But the continuity equation gives:

$$\frac{1}{A}\frac{dA}{dx} + \frac{1}{V}\frac{dV}{dx} + \frac{1}{\rho}\frac{d\rho}{dx} = 0$$

But using the information supplied:

$$\rho = \frac{p}{RT} = \frac{150000}{287 \times 308} = 1.697 \text{ kg/m}^3$$

Therefore the continuity equation gives:

$$1 + \frac{1}{250}\frac{dV}{dx} + \frac{1}{1.697}\frac{d\rho}{dx} = 0 \qquad (1)$$

It is next noted that the conservation of momentum equation gives:

$$-\frac{1}{\rho}\frac{dp}{dx} = V\frac{dV}{dx} \quad , \text{ i.e.,} \quad \frac{dp}{dx} = -\rho V\frac{dV}{dx}$$

hence:

$$\frac{dp}{dx} = -1.697 \times 250 \times \frac{dV}{dx} = -424.3\,\frac{dV}{dx} \tag{2}$$

The conservation of energy equation gives:

$$c_p\frac{dT}{dx} + V\frac{dV}{dx} = 0$$

hence again assuming that c_p is equal to 1007 J / kg °C it follows that:

$$1007\,\frac{dT}{dx} + 250\,\frac{dV}{dx} = 0$$

i.e.:

$$\frac{dT}{dx} = -0.243\,\frac{dV}{dx} \tag{3}$$

Lastly, it is noted that from the perfect gas law, $p = \rho R T$, it follows that:

$$\frac{1}{p}\frac{dp}{dx} = \frac{1}{\rho}\frac{d\rho}{dx} + \frac{1}{T}\frac{dT}{dx}$$

i.e.:

$$\frac{1}{150000}\frac{dp}{dx} = \frac{1}{1.697}\frac{d\rho}{dx} + \frac{1}{308}\frac{dT}{dx} \tag{4}$$

Using eq. (3), eq. (4) becomes:

$$\frac{1}{150000}\frac{dp}{dx} = \frac{1}{1.697}\frac{d\rho}{dx} - \frac{0.2483}{308}\frac{dV}{dx}$$

i.e.:

$$\frac{dp}{dx} = 88915\,\frac{d\rho}{dx} - 120.9\,\frac{dV}{dx} \tag{5}$$

Substituting from eqs. (1) and (2) into eq. (5) then gives:

$$-424.3\,\frac{dV}{dx} \;=\; -88915 \times 1.697 \times\left(1 + \frac{1}{250}\,\frac{dV}{dx}\right) - 120.9\,\frac{dV}{dx}$$

i.e.:

$$\frac{dV}{dx}\left(-424.3 + 120.9 + \frac{88915 \times 1.697}{250}\right) \;=\; -88915 \times 1.697$$

Hence:

$$\frac{dV}{dx} \;=\; 502.6\ \mathrm{s}^{-1} \tag{6}$$

Using this result in eq. (1) then gives:

$$1 \;+\; \frac{502.6}{250} \;+\; \frac{1}{1.697}\,\frac{d\rho}{dx} \;=\; 0$$

i.e.:

$$\frac{d\rho}{dx} \;= -1.687 \;+\; \frac{1.697 \times 502.6}{250} \;= 1.715\ \mathrm{kg/m^4}$$

Similarly, using eq. (6) in eq. (2) gives:

$$\frac{dp}{dx} \;=\; 424.3 \times 502.6 \;=\; 213250\ \mathrm{Pa/m}$$

Therefore the values of *dp/dx* , *dV/dx* and *dρ/dx* are 213250 Pa/m, -502.6 m/s per m, and 1.715 kg/m^4 respectively.

PROBLEM 2.6

Consider an isothermal air flow through a duct. At a certain section of the duct, the velocity, temperature and pressure are 200 m/s, 25°, and 120 kPa respectively. If the velocity is decreasing at this section at a rate of 30 per cent per m find *dp/dx, ds/dx* and *dp/dx.*

SOLUTION

The flow considered in this problem is not adiabatic.

The conservation of momentum equation gives:

$$-\frac{1}{\rho}\frac{dp}{dx} = V\frac{dV}{dx}\ , \text{ i.e.,}\quad \frac{dp}{dx} = -\rho V\frac{dV}{dx}$$

But:

$$\rho = \frac{p}{RT} = \frac{120000}{287 \times 298} = 1.403 \text{ kg/m}^3$$

and:

$$\frac{1}{V}\frac{dV}{dx} = -0.3$$

so:

$$\frac{dp}{dx} = +1.403 \times 200 \times 200 \times 0.3 = 16836 \text{ Pa/m}$$

Because the flow is isothermal, the perfect gas law, $p = \rho R T$, gives:

$$\frac{1}{p}\frac{dp}{dx} = \frac{1}{\rho}\frac{d\rho}{dx}$$

Hence:

$$\frac{d\rho}{dx} = \frac{\rho}{p}\frac{dp}{dx} = \frac{1.403}{120000}\times 16836 = 0.1968 \text{ kg/m}^3\text{/m}$$

Lastly since:

$$\frac{1}{c_{\mathrm{p}}}\frac{ds}{dx} = \frac{1}{T}\frac{dT}{dx} - \left(\frac{\gamma-1}{\gamma}\right)\frac{1}{p}\frac{dp}{dx}$$

it follows that for the isothermal situation being considered:

$$\frac{1}{c_{\mathrm{p}}}\frac{ds}{dx} = -\left(\frac{\gamma-1}{\gamma}\right)\frac{1}{p}\frac{dp}{dx}$$

which gives:

$$\frac{1}{1007}\frac{ds}{dx} = -\left(\frac{0.4}{1.4}\right)\times\frac{1}{120000}\frac{dp}{dx}$$

i.e.:

$$\frac{ds}{dx} = -\left(\frac{0.4}{1.4}\right)\times\frac{1007}{120000}\times 16836 = -40.36 \text{ J/kg-K per m}$$

The entropy is changing because of the heat transfer at the wall.

Therefore, the values of *dp/dx, dp/dx*, and *ds/dx* are 16.84 kPa/m, 0.1968 kg/m^3 / m, and - 40.36 J / kg-K per m respectively.

PROBLEM 2.7

Consider adiabatic air flow through a variable area duct. At a certain section of the duct, the flow area is 0.1 m^2, the pressure is 120 kPa, and the temperature is 15°C and the duct area is changing at a rate of 0.1 m^2/m. Plot the variations of *dp/dx, dV/dx* and *dρ/dx* with the velocity at the section for velocities between 50 m/s and 300 m/s.

SOLUTION

The continuity equation gives:

$$\frac{1}{A}\frac{dA}{dx} + \frac{1}{V}\frac{dV}{dx} + \frac{1}{\rho}\frac{d\rho}{dx} = 0$$

But using the information supplied:

$$\rho = \frac{p}{RT} = \frac{120000}{287 \times 288} = 1.452 \text{ kg/m}^3$$

Therefore the continuity equation gives:

$$\frac{1}{0.1} \times 0.1 + \frac{1}{V}\frac{dV}{dx} + \frac{1}{1.452}\frac{d\rho}{dx} = 0 \tag{1}$$

It is next noted that the conservation of momentum equation gives:

$$-\frac{1}{\rho}\frac{dp}{dx} = V\frac{dV}{dx} \quad , \quad \text{i.e.,} \quad \frac{dp}{dx} = -\rho V\frac{dV}{dx}$$

hence:

$$\frac{dp}{dx} = -1.452V\frac{dV}{dx} \tag{2}$$

The conservation of energy equation gives:

$$c_p\frac{dT}{dx} + V\frac{dV}{dx} = 0$$

hence again assuming that c_p is equal to 1007 J / kg °C it follows that:

$$1007 \frac{dT}{dx} + V \frac{dV}{dx} = 0 \tag{3}$$

Lastly, it is noted that from the perfect gas law, $p = \rho R T$, it follows that:

$$\frac{1}{p} \frac{dp}{dx} = \frac{1}{\rho} \frac{d\rho}{dx} + \frac{1}{T} \frac{dT}{dx}$$

i.e.:

$$\frac{1}{120000} \frac{dp}{dx} = \frac{1}{1.452} \frac{d\rho}{dx} + \frac{1}{288} \frac{dT}{dx} \tag{4}$$

Using eq. (3), eq. (4) becomes:

$$\frac{1}{120000} \frac{dp}{dx} = \frac{1}{1.452} \frac{d\rho}{dx} - \frac{1}{1007 \times 288} V \frac{dV}{dx}$$

i.e.:

$$\frac{dp}{dx} = 82645 \frac{d\rho}{dx} - 0.4138 V \frac{dV}{dx} \tag{5}$$

Substituting from eqs. (1) and (2) into eq. (5) then gives:

$$-1.452 V \frac{dV}{dx} = -82645 \times 1.452 \times \left(1 + \frac{1}{V} \frac{dV}{dx}\right) - 0.4138 V \frac{dV}{dx}$$

i.e.:

$$\frac{dV}{dx}\left(0.4138V + 1.452V + \frac{82645 \times 1.452}{V}\right) = -82645 \times 1.452$$

i.e.:

$$\frac{dV}{dx} = -\frac{120000}{120000/V - 1.038V} \tag{6}$$

Using this result in eq. (1) then gives:

$$\frac{1}{1.452}\frac{d\rho}{dx} = -1 + \frac{120000}{120000 - 1.038V^2}$$

i.e.:

$$\frac{d\rho}{dx} = -1.452 + \frac{174240}{120000 - 1.038V^2} \tag{7}$$

Similarly, using eq. (6) in eq. (2) gives:

$$\frac{dp}{dx} = \frac{124240V}{120000/V - 1.038V}$$

For any value of *V* (*m/s*), eqs. (6), (7) and (8) allow *dV/dx* (1 / m*), dρ/dx* (Kg / m^3 / m*)*, and *dp/dx* (Pa / m) to be found. Some results are given in the following table, these results also being shown in Figs. 2.7a, 2.7b and 2.7c that follow the table.

***V* – m/s**	***dp/dx* – Pa/m**	***dρ/dx* - Kg / m^3 / m**	***dV/dx* - 1 / m**
50	3710	0.0321	-51.1
75	8585	0.0742	-78.8
100	15894	0.1374	-109.5
125	26230	0.2268	-144.5
150	40559	0.3506	-186.2
175	60480	0.5229	-238.0
200	88780	0.7675	-305.7
225	130716	1.1305	-400.1
250	197417	1.7077	-543.9
275	317159	2.7418	-794.3
300	588781	5.0900	-1315.7

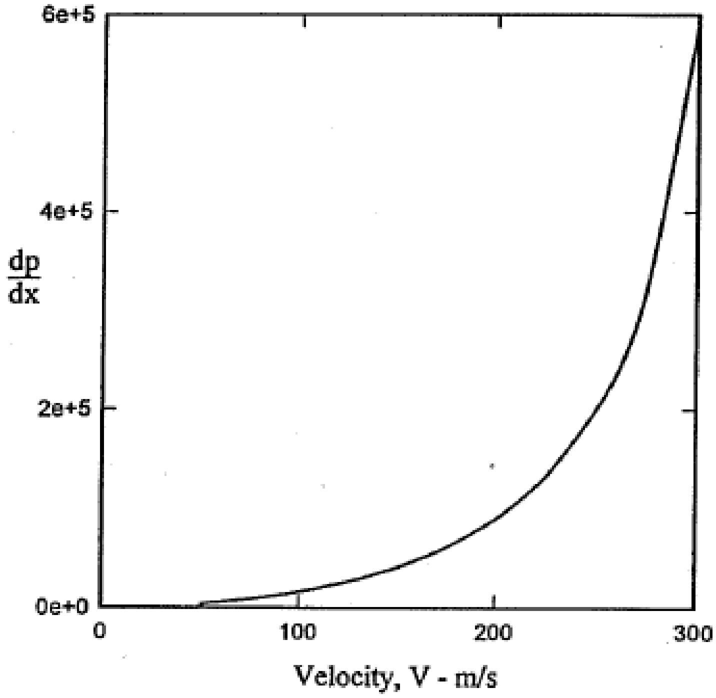

Figure P2.7a

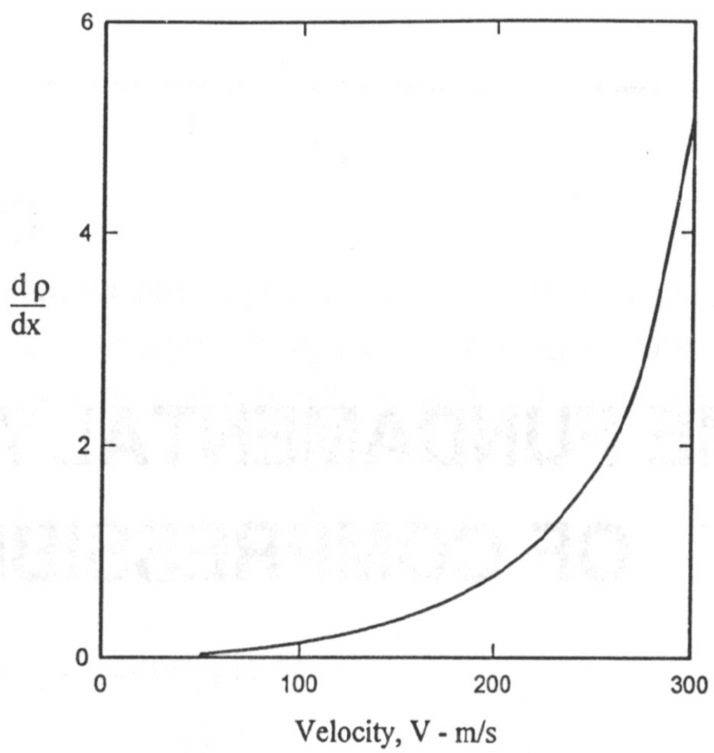

Figure P2.7b

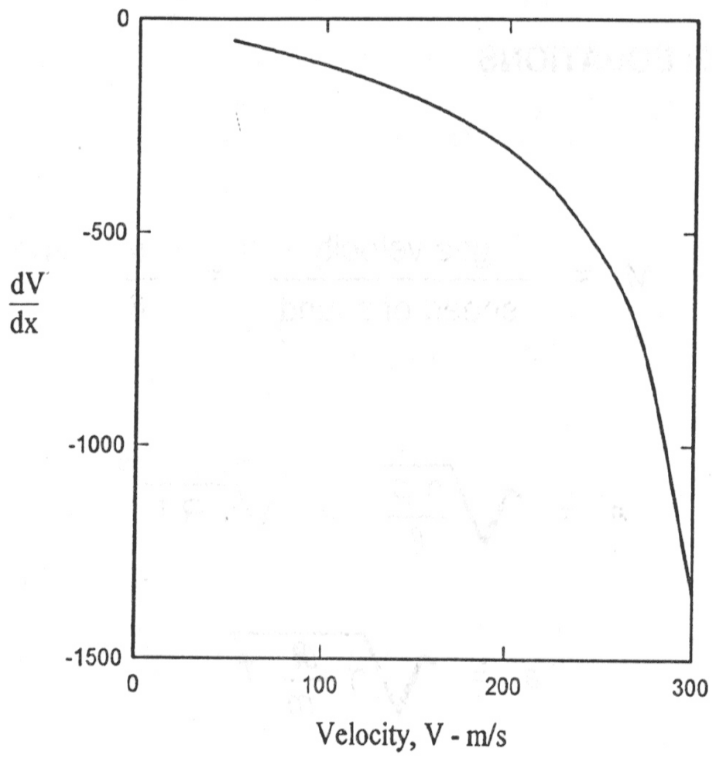

Figure P2.7c

PROBLEM 2.8

Methane flows through a circular pipe which has a diameter of 4cm. The temperature, pressure, and velocity at the inlet to the pipe are 200 K, 250 kPa, and 30 m/s respectively. Assuming that the flow is steady and isothermal calculate the pressure on the exit plane and the heat added to the methane in the pipe if the velocity on the pipe exit plane is 35m/s. Assume that the methane can be treated as a perfect gas with a specific heat ratio of 1.32 and a molar mass of 16.

SOLUTION

As shown in the textbook momentum conservation considerations give:

$$-\frac{dp}{\rho} = V\,dV$$

Using the perfect gas law this can be written as:

$$-RT\frac{dp}{p} = V\,dV$$

For an isothermal flow this can be integrated between any two points, 1 and 2, in the flow to give:

$$\frac{V_2^2}{2} - \frac{V_1^2}{2} = RT\left(\ln p_2 - \ln p_1\right) = RT\ln\left(\frac{p_2}{p_1}\right) \;,\quad \text{i.e.,}\quad \ln\left(\frac{p_2}{p_1}\right) = \frac{1}{RT}\left(\frac{V_2^2}{2} - \frac{V_1^2}{2}\right)$$

Using the information provided:

$$R = \frac{8314}{16} = 519.6 \;\; \text{J/kg K}$$

Hence

$$\ln\left(\frac{p_2}{p_1}\right) = \frac{1}{519.6 \times 200}\left(\frac{35^2}{2} - \frac{30^2}{2}\right) = 0.001564 \;,\quad \text{i.e.,}$$

$$p_2 = p_1\,e^{0.001564} = 250e^{0.001564} = 250.4 \;\; \text{kPa}$$

Now since using the perfect gas law gives:

$$\rho = \frac{p}{RT}\ ,\ i.e.\quad \rho_1 = \frac{p_1}{RT_1} = \frac{250000}{519.6 \times 200} = 2.406\ \text{kg/m}^3$$

Therefore the mass flow rate through the pipe is given by:

$$\dot{m} = \rho_1 V_1 A = 2.406 \times 30 \times \frac{\pi}{4} \times 0.04^2 = 0.091\ \text{kg/s}$$

The heat added per unit mass of air flow is given by the energy equation as:

$$q = \left(c_p T_2 + \frac{V_2^2}{2} \right) - \left(c_p T_1 + \frac{V_1^2}{2} \right)$$

i.e., since isothermal flow is being considered:

$$q = \frac{V_2^2}{2} - \frac{V_1^2}{2} = \frac{35^2}{2} - \frac{30^2}{2} = 162.5\ \text{J/kg}$$

Hence the rate of heat addition to the flow is given by:

$$Q = \dot{m} q = 0.091 \times 162.5 = 14.79\ \text{J/s}$$

Therefore the exit plane pressure is 250.4 kPa and the rate heat is added to the flow is 14.79 J/s.
