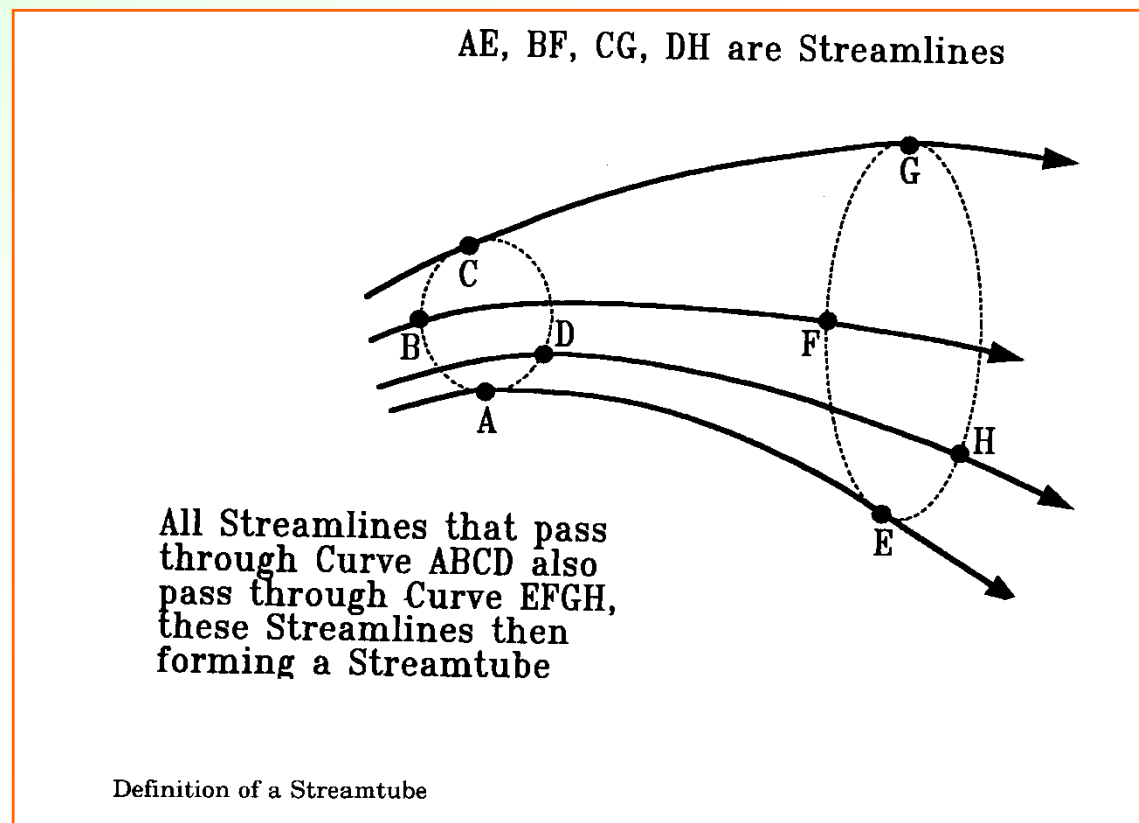# Chapter Two

THE EQUATIONS OF STEADY ONE- DIMENSIONAL COMPRESSIBLE FLUID FLOW

INTRODUCTION

Many of the compressible flows that occur in engineering practice can be adequately modeled as a steady flow (i.e., not changing with time) through a duct or streamtube whose cross-sectional area is changing relatively slowly in the flow direction. A duct is here taken to mean a solid walled channel while a streamtube is defined by considering a closed curve drawn in a fluid flow.

A series of streamlines will pass through this curve as shown in the following figure.

AE, BF, CG, DH are Streamlines

All Streamlines that pass through Curve ABCD also pass through Curve EFGH, these Streamlines then forming a Streamtube
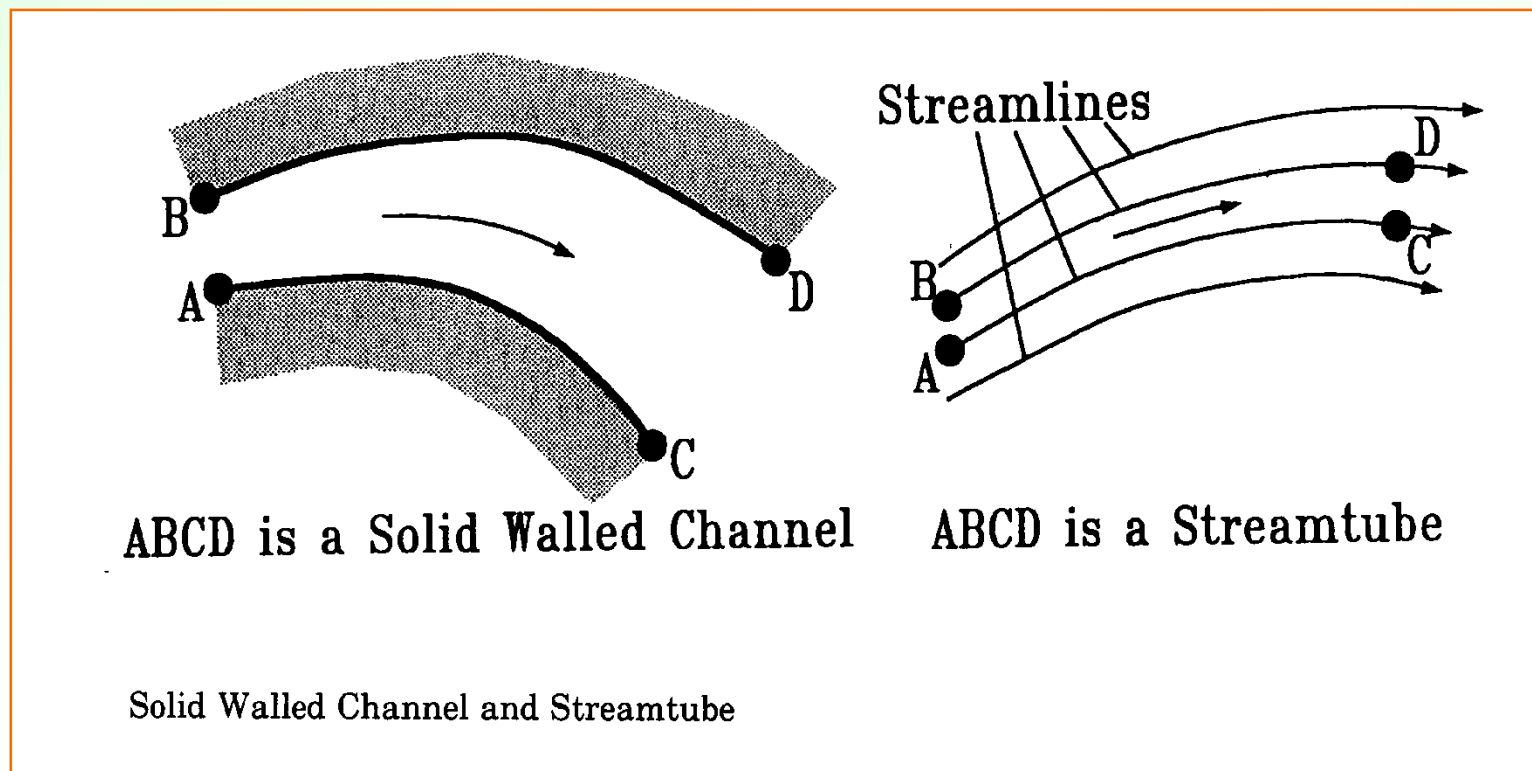
Definition of a Streamtube

Compressible Fluid Flow

Further downstream, these streamlines can be joined by another curve as shown in the figure. Since there is no flow normal to a streamline, in steady flow the rate at which fluid crosses the area defined by the first curve is equal to the rate fluid at which crosses the area defined by the second curve. The streamlines passing through the curves, therefore, effectively define the "walls" of a duct and this "duct" is called a streamtube. Of course, in the case of a duct with solid walls, streamlines lie along the walls and the duct is, effectively, also a streamtube.
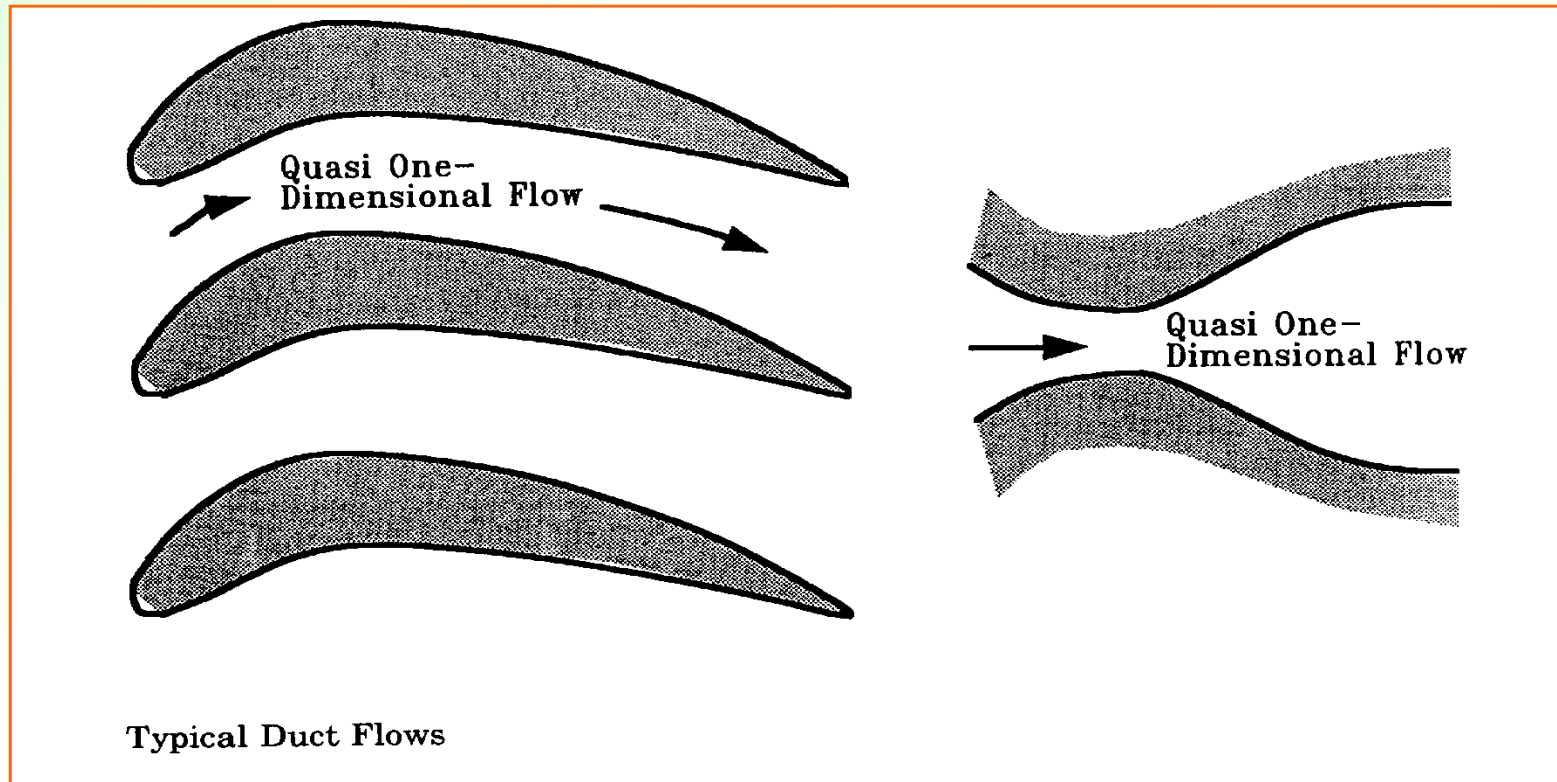
In the case of both flow through a streamtube and flow through a solid-walled duct, there can be no flow through the "walls" of the duct, there being no flow through a solid wall and, by definition, no flow normal to a streamline. The two types of duct are shown in the following figure.

ABCD is a Solid Walled Channel   ABCD is a Streamtube

Solid Walled Channel and Streamtube

Compressible Fluid Flow

Example of the type of flow being considered in this chapter are those through the blade passages in a turbine and the flow through a nozzle fitted to a rocket engine, these being shown in the following figure.
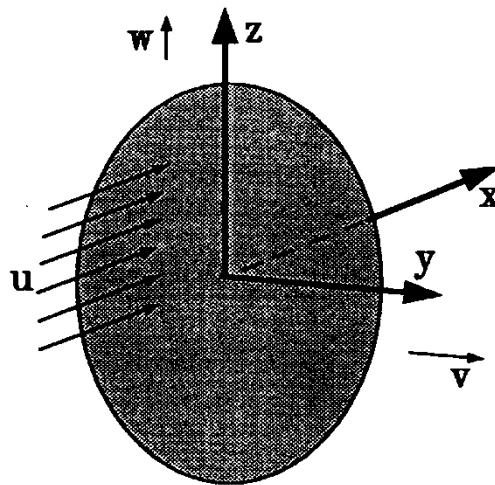

Quasi One-
Dimensional Flow

Quasi One-
Dimensional Flow

Typical Duct Flows

Compressible Fluid Flow

In many such practical situations it is adequate to assume that the flow is steady and one-dimensional. As discussed in the previous chapter, steady flow implies that none of the properties of the flow are varying. with time. In most real flows that are steady on the average, the instantaneous values of the flow properties, in fact, fluctuate about mean the values. However, an analysis of such flows based on the assumption of steady flow usually gives a good description of the mean values of the flow variables.
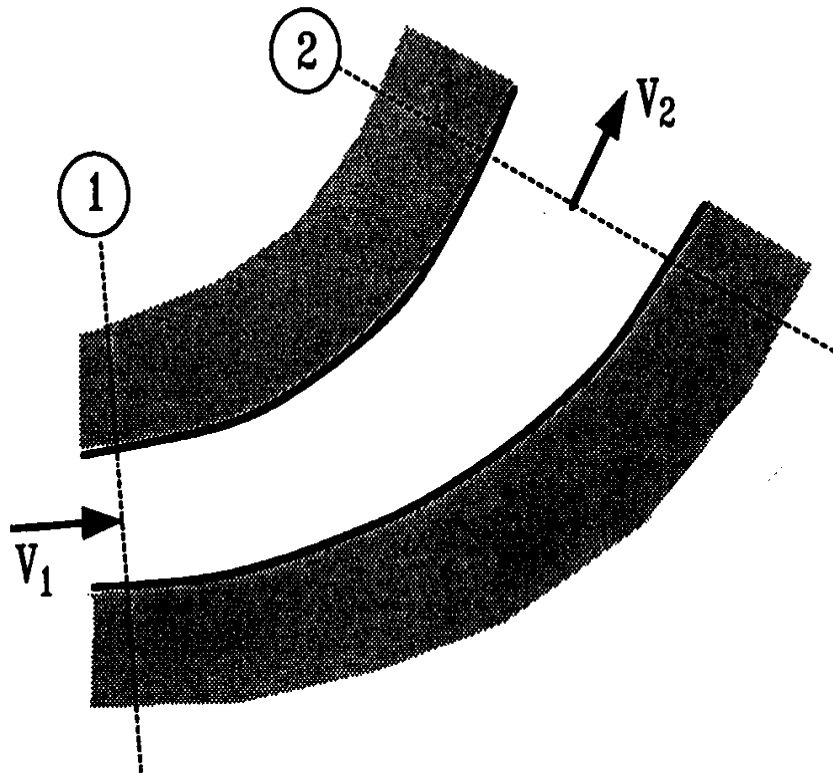
One-dimensional flow is, strictly, flow in which the reference axes can be so chosen that the velocity vector has only one component over the portion of the flow field considered, i.e. if u, v and w are the x, y and z components of the velocity vector then, strictly, for the flow to be one-dimensional it is necessary that it be possible for the x direction to be so chosen that the velocity components v and w are zero (see figure).

One-Dimensional Flow

In a one-dimensional flow the velocity at a section of the duct will here be given the symbol V as indicated in the following figure.
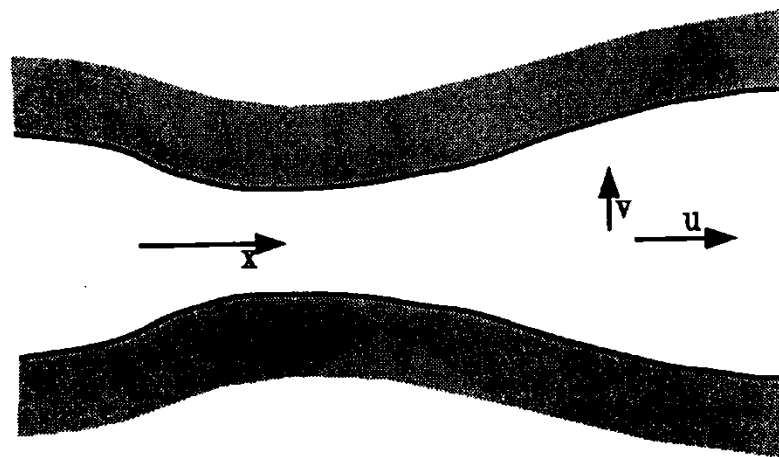

Definition of a Velocity V

Compressible Fluid Flow

Strictly speaking, the equations of one-dimensional flow are only applicable to flow in a straight pipe or stream tube of constant area. However, in many practical situations, the equations of one-dimensional flow can be applied with acceptable accuracy to flows with a variable area provided that the rate of change of area and the curvature are small enough for one component of the velocity vector to remain dominant over the other two components.

Compressible Fluid Flow

For example, although the flow through a nozzle of the type shown in the following figure is not strictly one-dimensional, because v remains very much less than u, the flow can be calculated with sufficient accuracy for most purposes by ignoring v and assuming that the flow is one-dimensional i.e. by only considering the variation of u with x.

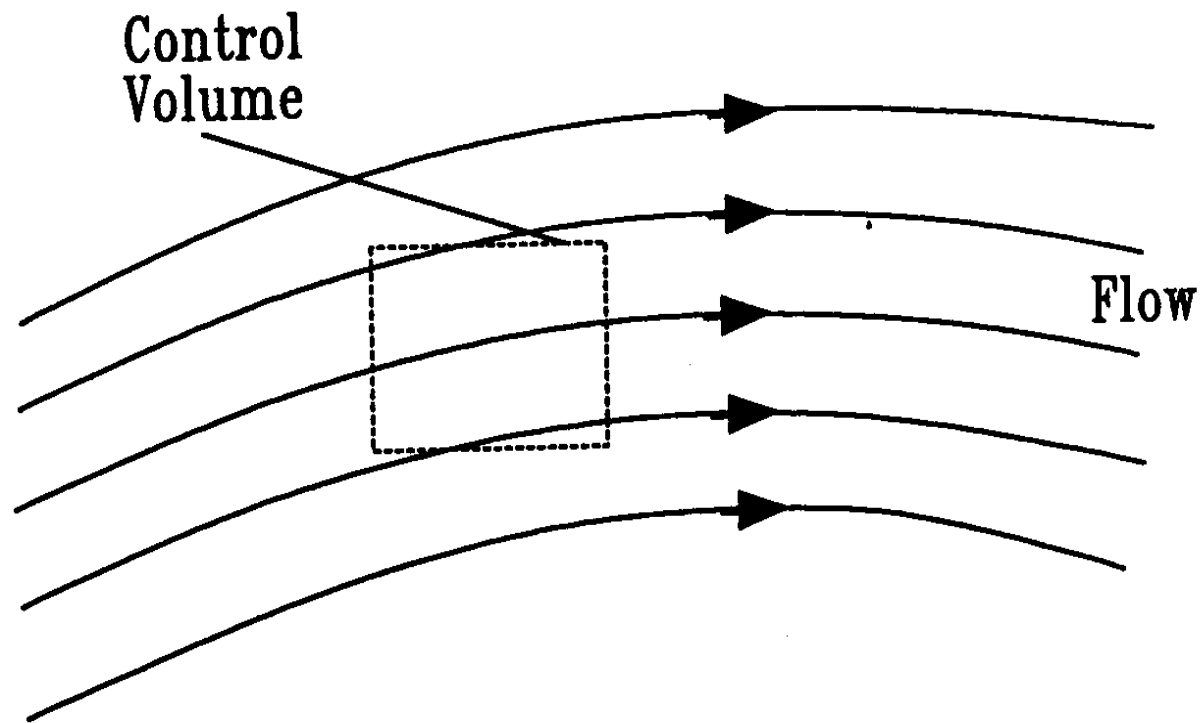Flow Situation That Can be Modelled as One-Dimensional

Compressible Fluid Flow

Such flows in which the flow area is changing but in which the flow at any section can be treated as one-dimensional, are commonly referred to as "quasi one-dimensional" flows.

CONTROL VOLUME

The concept of a control volume is used in the derivation and application of many equations of compressible fluid flow. As discussed in the previous chapter, a control volume is an arbitrary imaginary volume fixed relative to the coordinate system being used (the coordinate system can be moving) and bounded by a control surface through which fluid may pass as shown in the following figure.
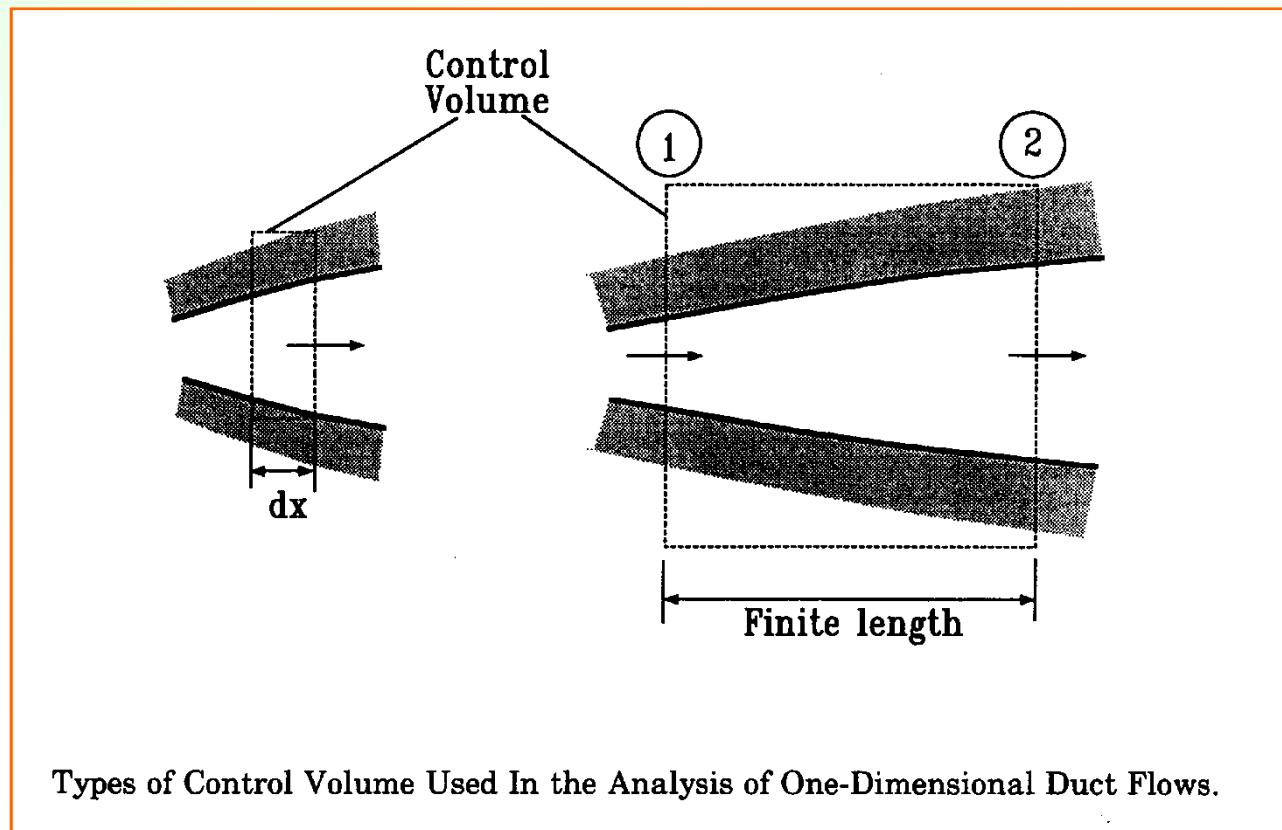
Compressible Fluid Flow

Control Volume in a General Two-Dimensional Flow.

Compressible Fluid Flow

In applying the control volume concept, the effects of forces on the control surface and mass and energy transfers through this surface are considered. In general, it should be noted, it is possible for conditions in the control volume to be changing with time but for the reasons mentioned above attention will here be restricted to steady flow in which conditions inside and outside the control volume are constant in time in terms of the coordinate system being used.

In the case of one-dimensional duct flow that is here being considered, control volumes of the type shown in the following figure are used. These control volumes either cover a differentially short length *dx* of the duct or a finite length of the duct as shown in the figure.

Types of Control Volume Used In the Analysis of One-Dimensional Duct Flows.
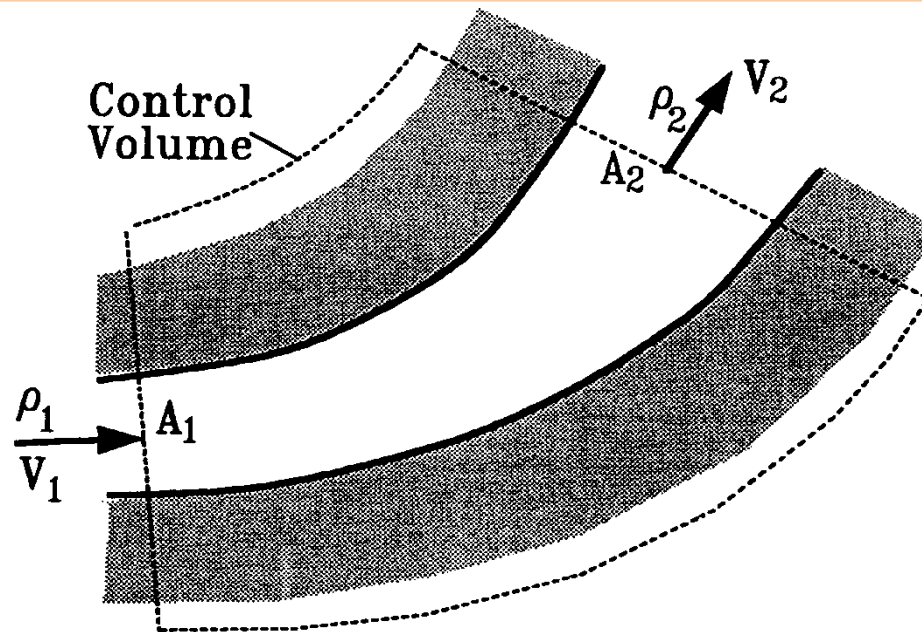
In the case of the differentially short control volume, the changes in the flow variables through the control volume, such as those in velocity and pressure i.e. *d V* and *d p* , will also be small and in the analysis of the flow the products of these differentially small changes such as *d V* x *d p* will be neglected.

CONTINUITY EQUATION: The continuity equation is obtained by applying the principle of conservation of mass to flow through a control volume. Consider the situation shown in the following figure. The changes through this control volume are indicated in this figure, it being recalled that one-dimensional flow is being considered.

Control Volume Used In Derivation of Continuity Equation.
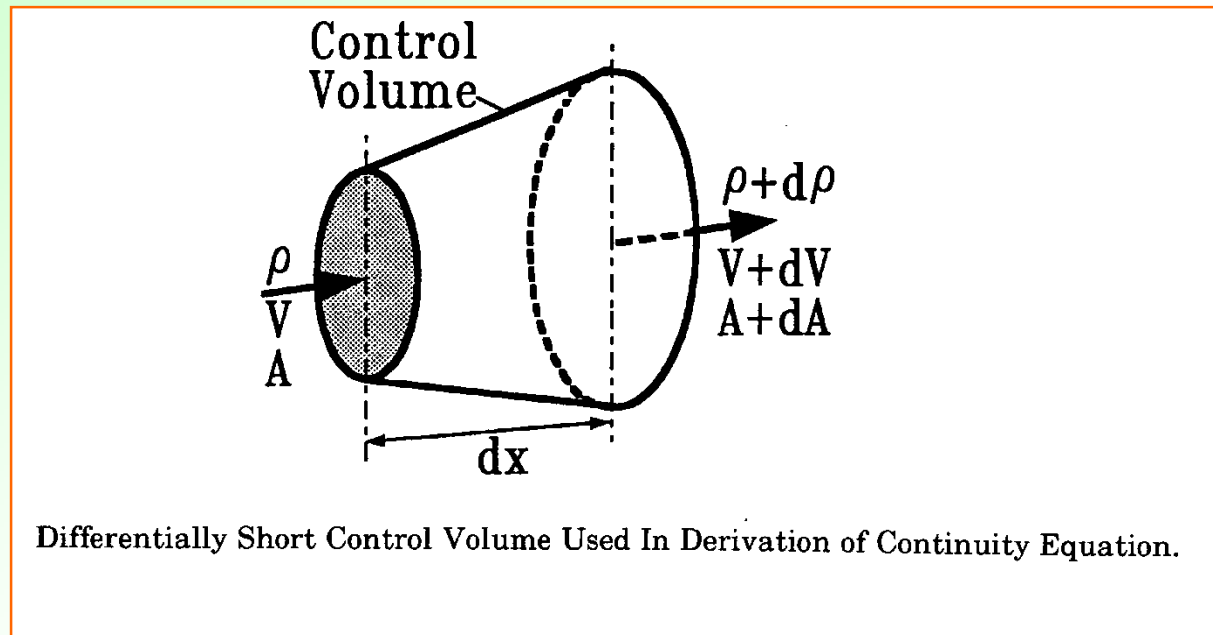
Compressible Fluid Flow

Since there is no mass transfer across the walls of the stream tube, the only mass transfer occurs through the ends of the control volume. If the possibility of a source of mass within the control volume is excluded, the principle of conservation of mass requires that the rate at which mass enters through the left hand face of the control volume be equal to the rate at which mass leaves through the right hand face of the control volume i.e. that:

$$\dot{m}_1 = \dot{m}_2$$

Since the rate at which mass crosses any section of the duct, i.e., $\dot{m}$ is equal to $\rho\, V A$ where A is the cross-sectional area of the duct at the section considered, the above equation gives:

$$\rho_1 V_1 A_1 = \rho_2 V_2 A_2$$

Compressible Fluid Flow

Differentially Short Control Volume Used In Derivation of Continuity Equation.

For the differentially short control volume shown in the above figure, this equation gives:

$$\rho VA = (\rho + d\rho)(V + dV)(A + dA)$$

i.e., neglecting higher order terms as discussed above:

$$VAd\rho + \rho AdV + \rho VdA = 0$$
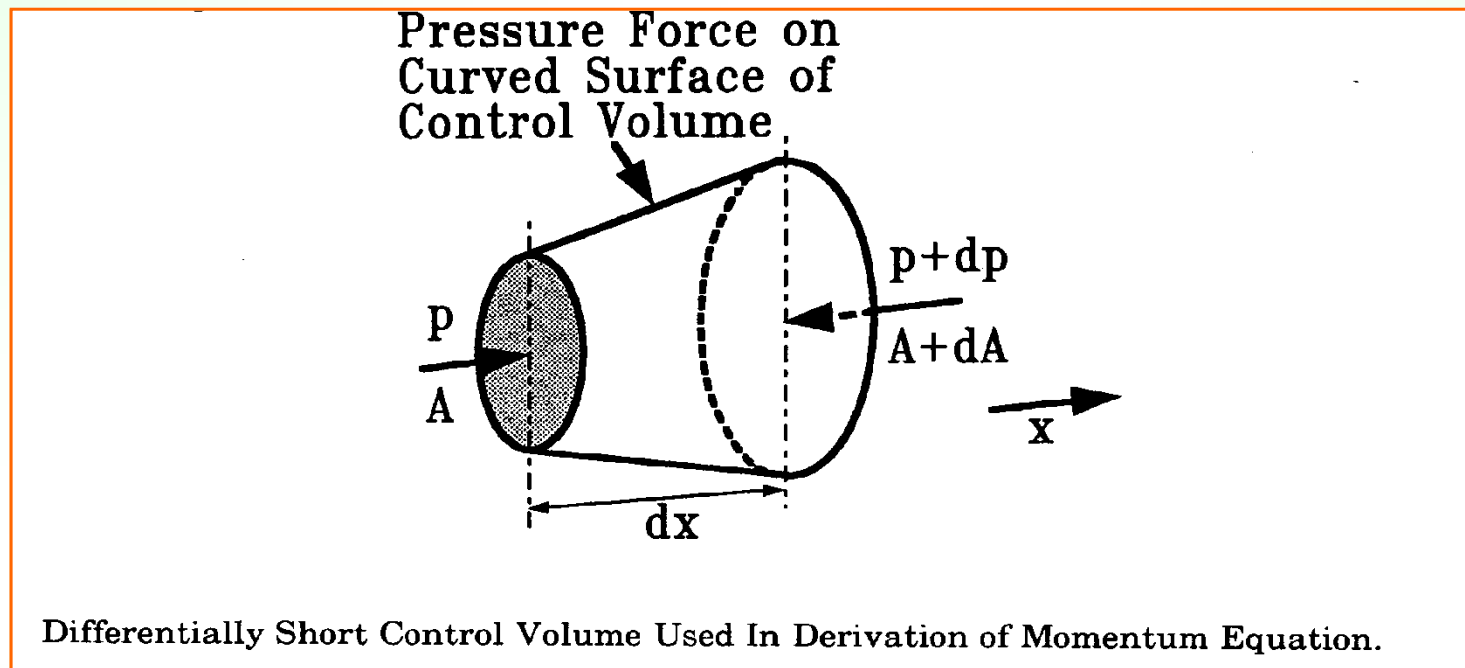
Compressible Fluid Flow

Dividing this equation by ρVA then gives:

$$\frac{d\rho}{\rho} + \frac{dV}{V} + \frac{dA}{A} = 0$$

This equation relates the fractional changes in density, velocity and area over a short length of the control volume. If the density can be assumed constant, this equation indicates that the fractional changes in velocity and area have opposite signs, i.e. if the area increases the velocity will decrease and vice versa. However, the equation indicates that in compressible flow, where the fractional change in density is significant, no such simple relation between area and velocity changes exists.

MOMENTUM EQUATION (EULER'S EQUATION): Euler's equation is obtained by applying conservation of momentum to a control volume which again consists of a short length, *dx*, of a stream tube. Steady flow is again assumed. The forces acting on the control volume are shown in the following figure:

Differentially Short Control Volume Used In Derivation of Momentum Equation.

Compressible Fluid Flow

Because the flow is steady, conservation of momentum requires that for this control volume the net force in direction x be equal to the rate at which momentum leaves the control volume in the x direction minus the rate at which it enters in the x-direction since the flow is steady. Since, by the fundamental assumptions previously listed, gravitational forces are being neglected the only forces acting on the control volume are the pressure forces and the frictional force exerted on the surface of the control volume. Thus, the net force on the control volume in the x-direction is:

$$pA - (p + dp)(A + dA) + \frac{1}{2}(p + p + dp)\left[(A + dA) - A\right] - dF_\mu$$

The term dF_μ is the frictional force.

Compressible Fluid Flow

Rearranging the above equation then gives the net force on the control volume in the *x*-direction as:

$$- A dp \ - \ dF_{\mu}$$

In writing this equation, the higher order terms such as *d p x d A* have again been neglected since *d x* is taken to be small.

The rate at which momentum crosses any section of the duct is equal to $\dot{m} V$, the difference between the rate at which momentum leaves the control volume and the rate at which momentum enters the control volume is given by:

$$\rho \, V \, A \, [\, (V \ + \ d \, V) \, - \, V] = \rho \, V \, A \, d \, V$$

since no momentum enters through the curved walls of the control volume.

Compressible Fluid Flow

Since conservation of momentum requires that the net force on the control volume be equal to the rate at which momentum leaves the control volume minus the rate at which it enters the control volume, the above equations give:

$$- A dp \; - \; dF_{\mu} = \rho \, V \, A \, d \, V$$

As discussed in the previous chapter, viscous friction effects will be neglected in the initial portion of this course i.e. the term dF_{μ} in the above equation is assumed to be negligible. In this case, the equation can be rearranged to give:

$$- dp = \rho \, V \, d \, V$$

This equation, i.e.:

$$-dp = \rho \, V \, d \, V$$

is Euler's equation for steady flow through a duct. Since V is, by the choice of the x-direction, always positive, i.e. the x-direction is taken in the direction of the flow, this equation indicates that dp and dV are opposite in sign, i.e. that an increase in velocity is always associated with a decrease in pressure and vice versa. This is an obvious result because the decrease in pressure is required to generate the force needed to accelerate the flow, i.e. to increase the velocity, and vice versa.

Compressible Fluid Flow

If Euler's equation is integrated in the *x*-direction along the streamtube, it gives:

$$\frac{V^2}{2} + \int \frac{dp}{\rho} = \text{constant}$$

In order to evaluate the integral, the variation of density with pressure must be known. If the flow can be assumed to be incompressible, i.e. if the density can be assumed constant, this equation gives:

$$\frac{V^2}{2} + \frac{p}{\rho} = \text{constant} \quad \text{or} \quad \frac{\rho V^2}{2} + p = \text{constant}$$

which is, of course, Bernoulli's equation. It should, therefore, be clearly understood that Bernoulli's equation only apples in incompressible flow.

STEADY FLOW ENERGY EQUATION: **This states that, for flow through the type of control volume considered above, if the fluid enters at section 1 with velocity V_1 and with enthalpy h_1 per unit mass, and leaves through section 2 with velocity V_2 and enthalpy h_2 then:**

$$h_2 + \frac{V_2^2}{2} = h_1 + \frac{V_1^2}{2} + q - w$$

where q is the heat transferred into the control volume per unit mass of fluid flowing through it and w is the work done by the fluid per unit mass in flowing through the control volume. Attention will here be restricted to flows in which no work is done so that w is zero.

Further, since only calorically perfect gases are being considered in this chapter:

$$h = c_p T$$

Hence, the steady flow energy equation for the present purposes can be written as:

$$c_p T_2 + \frac{V_2^2}{2} = c_p T_1 + \frac{V_1^2}{2} + q - w$$

Compressible Fluid Flow

This equation indicates that in compressible flows, changes in velocity will, in general, induce changes in temperature and that heat addition can cause velocity changes as well as temperature changes.

If the flow is adiabatic, i.e. if there is no heat transfer to or from the flow, the above equations give:

$$c_p T_2 + \frac{V_2^2}{2} = c_p T_1 + \frac{V_1^2}{2}$$

and:

$$c_p dT + V dV = 0$$

This equation shows that in adiabatic flow, an increase in velocity is always accompanied by a decrease in temperature.

EQUATION OF STATE

When applied between any two points in the flow, this equation gives:

$$\frac{p_1}{\rho_1 T_1} = \frac{p_2}{\rho_2 T_2}$$

When applied between the inlet and the exit of a differentially short control volume, this equation becomes:

$$\frac{p}{\rho T} = \frac{p + dp}{(\rho + d\rho)(T + dT)}$$

Compressible Fluid Flow

Since dp/p, $d\rho/\rho$ and dT/T are small, when higher order terms are neglected this gives:

$$\frac{p}{\rho T} = \frac{p}{\rho T}(1 + \frac{dp}{p})(1 - \frac{d\rho}{\rho})(1 - \frac{dT}{T})$$

i.e.:

$$\frac{dp}{p} - \frac{d\rho}{\rho} - \frac{dT}{T} = 0$$

This equation shows how the changes in pressure, density and temperature are interrelated in compressible flows.

ENTROPY CONSIDERATIONS: In studying compressible flows, another variable, the entropy, generally has to be introduced. The entropy basically places limitations on which flow processes are physically possible and which are physically excluded. The entropy change between any two points in the flow is given by:

$$s_2 - s_1 = c_p \ln\left(\frac{T_2}{T_1}\right) - R \ln\left(\frac{p_2}{p_1}\right)$$

Since $R = c_p - c_v$, this equation can be written:

$$\frac{s_2 - s_1}{c_p} = \ln\left[\left(\frac{T_2}{T_1}\right)\left(\frac{p_2}{p_1}\right)^{-\left(\frac{\gamma-1}{\gamma}\right)}\right]$$

Compressible Fluid Flow

If there is no change in entropy, i.e. if the flow is isentropic, this equation requires that:

$$\frac{T_2}{T_1} = \left(\frac{p_2}{p_1} \right)^{\frac{\gamma - 1}{\gamma}}$$

Hence, since the perfect gas law gives:

$$\frac{T_2}{T_1} = \frac{p_2}{p_1} \frac{\rho_1}{\rho_2}$$

it follows that in isentropic flow:

$$\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1} \right)^{\gamma}$$

In isentropic flows, then, p / ρ^{γ} is a constant.

If the entropy equation is applied between the inlet and the exit of a differentially short control volume, it gives:

$$(s + ds) - s = c_p \ln\left(\frac{T + dT}{T}\right) - R \ln\left(\frac{p + dp}{p}\right)$$

Since, if ε is a small quantity, $\ln(1 + \varepsilon)$ is to first order equal to ε, the above equation gives:

$$ds = c_p \frac{dT}{T} - R \frac{dp}{p}$$

which can be written as:

$$\frac{ds}{c_p} = \frac{dT}{T} - \left(\frac{\gamma - 1}{\gamma}\right)\frac{dp}{p}$$

Compressible Fluid Flow

Lastly, it is noted that in an isentropic flow:

$$c_p dT = \frac{RT}{p} dp = \frac{dp}{\rho}$$

But the energy equation for isentropic flow, i.e. for flow with no heat transfer, gives:

$$c_p dT + VdV = 0$$
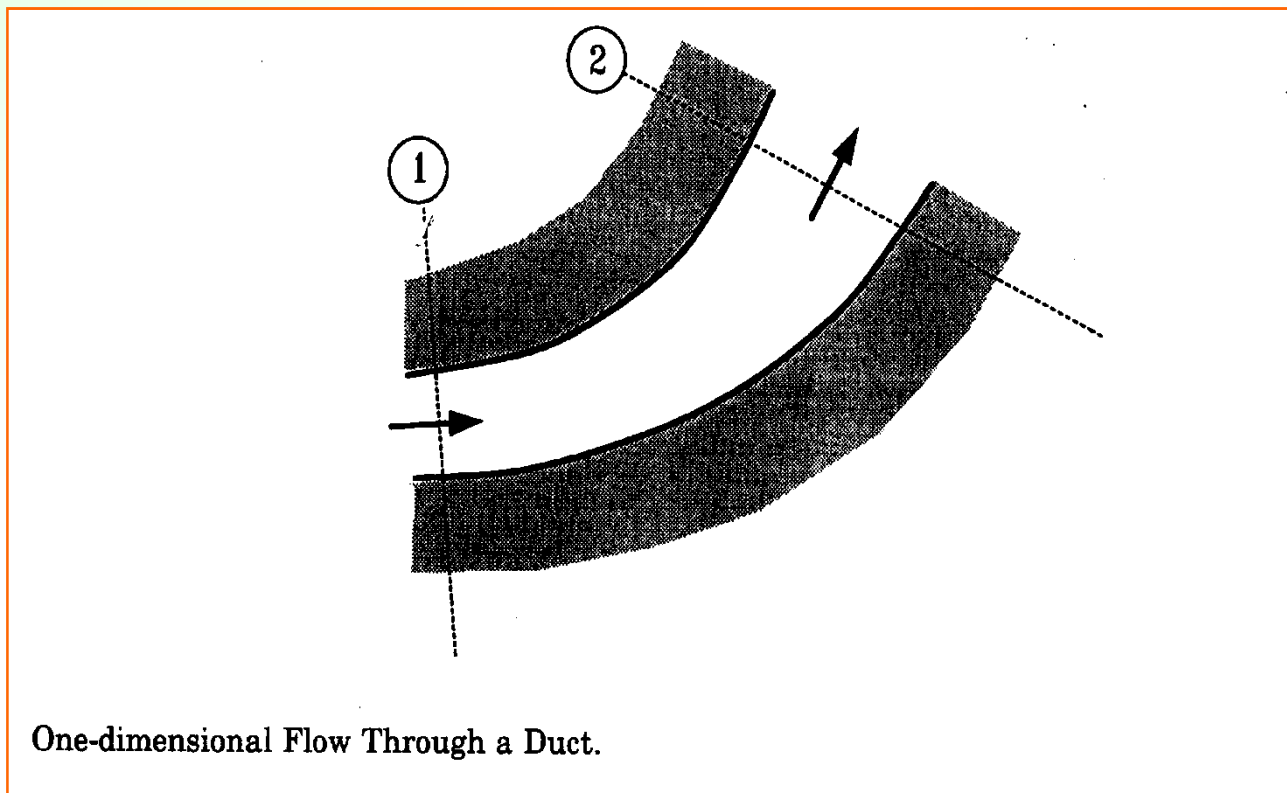
which can be written as:

$$\frac{dp}{\rho} + VdV = 0$$

This is identical to the result obtained using conservation of momentum considerations. In isentropic flow, then, it is not necessary to consider both conservation of energy and conservation of momentum since, when the "isentropic equation of state" is used, they give the same result.
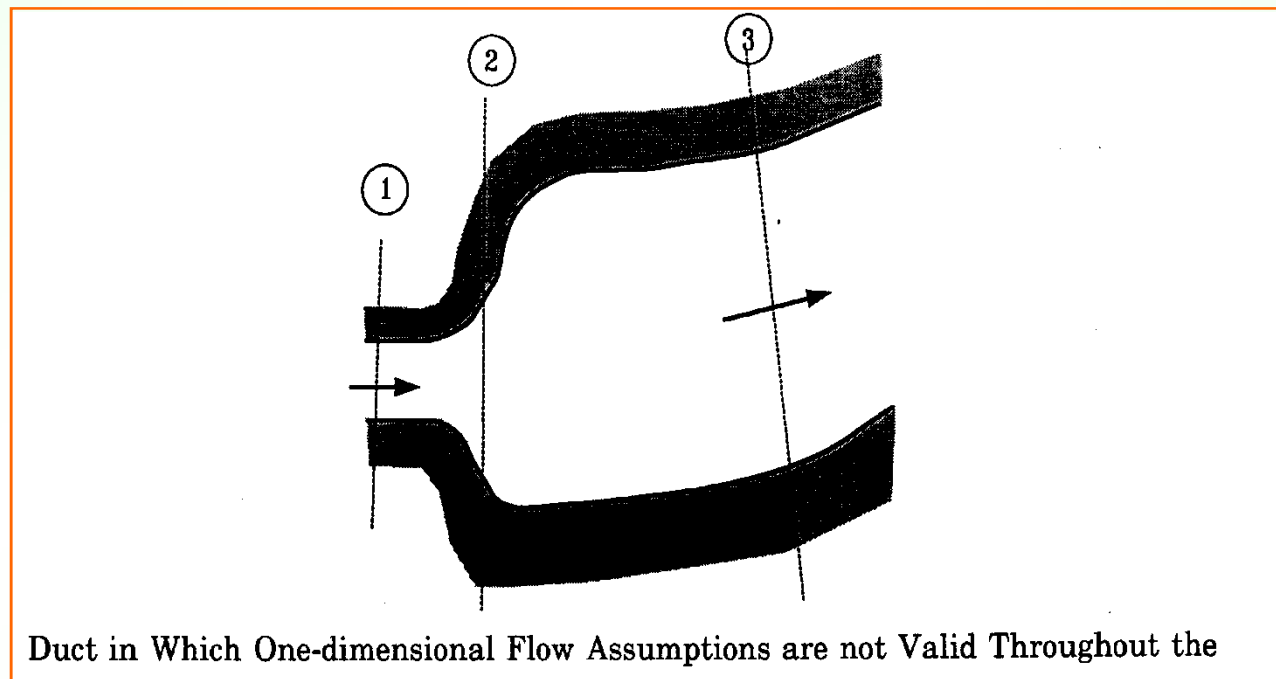
USE OF THE ONE-DIMENSIONAL FLOW EQUATIONS:
The most obvious application of the quasi-one-dimensional equations is to flow through a solid walled duct or a streamtube whose cross-sectional area is changing slowly with distance as shown in the following figure:

One-dimensional Flow Through a Duct.

Compressible Fluid Flow

For the one-dimensional flow assumption to be valid, the rate of change of duct area with respect to the distance x along the duct must remain small. However, in applying the one-dimensional flow equations to the flow through a duct, it should be noted that the flow does not have to be one-dimensional at all sections of the duct in order to use the one-dimensional flow equations. For example consider the following situation:
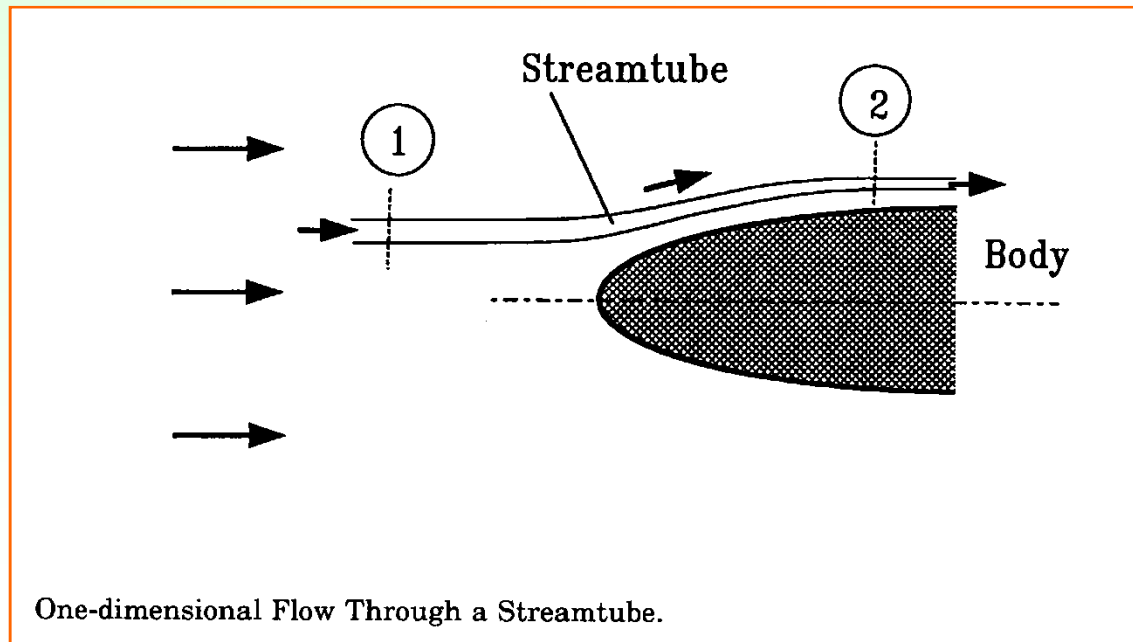

Duct in Which One-dimensional Flow Assumptions are not Valid Throughout the

Compressible Fluid Flow

The flow at section 2 cannot be assumed one dimensional. However, the conditions at sections 1 and 3 can be related by the one-dimensional equation.

The one-dimensional equations also, as discussed above, apply to the flow through any streamtube as discussed earlier. An example streamtube is shown in the following figure:

One-dimensional Flow Through a Streamtube.

The flow along the streamtube shown in this figure will be one-dimensional. As the fluid flows along this stream tube, its area changes and there are associated changes in the pressure, temperature and density

CONCLUDING REMARKS:

The equations discussed in the present chapter, while only being strictly applicable to flows that are one-dimensional, still form the basis of the analysis to an acceptable degree of accuracy of many compressible fluid flows that occur in engineering practice. The equations clearly indicate how, in compressible flows, changes in temperature and density are interlinked with changes in the velocity field.