Instructor's Solutions Manual

to

A Friendly Introduction to Analysis

Single and Multivariable

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Contents

Chapter 1 – Introduction

Section 1.1

1. **(a)** $(A \cup B) \cap C = \{1,2,3,4,5\} \cap \{1,8\} = \{1\}$.

   (b) $A \cup (B \cap C) = \{1,3,5\} \cup \{1\} = \{1,3,5\}$.

   (c) $(A \setminus C) \cup B = \{3,5\} \cup \{1,2,4\} = \{1,2,3,4,5\}$.

   (d) $(A \cap B) \times C = \{1\} \times \{1,8\} = \{(1,1),(1,8)\}$.

   (e) $C \times C = \{1,8\} \times \{1,8\} = \{(1,1),(1,8),(8,1),(8,8)\}$.

   (f) $\{\phi\} \cap A = \phi$.

2. Add equations $x - 2y = -4$ and $-2x + 2y = 2$ to get $-x = -2$. Thus, $x = 2$ and $y = 3$.

3. **(a)** $A \cap A = A$. Proof. If $x \in A \cap A$, then $x \in A$. Thus, $A \cap A \subseteq A$. Now, if $x \in A$, then $x \in A \cap A$. Thus, $A \subseteq A \cap A$. Hence, $A \cap A = A$.
   $A \cup A = A$. Proof is similar to the above.

   (b) $A \cap B = B \cap A$. Proof. If $x \in A \cap B$, then $x \in A$ and $x \in B$. Thus, $x \in B \cap A$ and so, $A \cap B \subseteq B \cap A$. Similarly, $B \cap A \subseteq A \cap B$. And hence, the equality holds.
   $A \cup B = B \cup A$. Proof is similar to the above.

   (c) $(A \cap B) \cap C = A \cap (B \cap C)$. Proof. If $x \in (A \cap B) \cap C$, then $x \in A \cap B$ and $x \in C$. Thus, $x \in A$, $x \in B$, and $x \in C$. Therefore, $x \in A$ and $x \in B \cap C$. Hence, $x \in A \cap (B \cap C)$. And therefore, $(A \cap B) \cap C \subseteq A \cap (B \cap C)$. Similarly, $A \cap (B \cap C) \subseteq (A \cap B) \cap C$. And hence, the equality follows.

   (d) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. Proof. If $x \in A \cup (B \cap C)$, then $x \in A$ or $x \in B \cap C$. Thus, $x \in A$ or $x \in B$, and $x \in A$ or $x \in C$. Therefore, $x \in A \cup B$ and $x \in A \cup C$. So $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$. Similarly, $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$. And the equality follows.

   (e) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$. Proof. If $x \in A \setminus (B \cap C)$, then $x \in A$ and $x \notin B \cap C$. Thus, $x \in A$ and $x \notin B$, or $x \in A$ and $x \notin C$. So $x \in (A \setminus B) \cup (A \setminus C)$ and $A \setminus (B \cap C) \subseteq (A \setminus B) \cup (A \setminus C)$. Similarly, $(A \setminus B) \cup (A \setminus C) \subseteq A \setminus (B \cap C)$. And the equality follows.

4. **(a)** Suppose $A \subseteq B$. We will show that $A \cap B \subseteq A$ and $A \subseteq A \cap B$. To this end, if $x \in A \cap B$, then $x \in A$ and $x \in B$. Clearly, $A \cap B \subseteq A$. If $x \in A$, since $A \subseteq B$ we have $x \in B$. Thus, $x \in A \cap B$ and so $A \subseteq A \cap B$. Hence, $A \cap B = A$. Conversely, suppose $A \cap B = A$ and $x \in A$. Then $x \in B$. And therefore, $A \subseteq B$.

   (b) If $x \in A \cap B$, then $x \in A$ and $x \in B$. So $x \notin A \setminus B$. Thus, $x \in A \setminus (A \setminus B)$. Hence, $A \cap B \subseteq A \setminus (A \setminus B)$. Conversely, if $x \in A \setminus (A \setminus B)$, then $x \in A$ and $x \notin A \setminus B$. Note that $x \in B$ since otherwise, if $x \notin B$, then since $x \in A$, we would have $x \in A \setminus B$, which is not the case. Hence, $x \in A \cap B$. And thus, $A \setminus (A \setminus B) \subseteq A \cap B$. And the equality follows.

   (c) If $x \in (A \setminus B) \cup (B \setminus A)$, then $x \in A \setminus B$ or $x \in B \setminus A$. Thus, $x \in A$ and $x \notin B$, or $x \in B$ and $x \notin A$. Therefore, $x \in A \cup B$ and $x \notin A \cap B$. So, $x \in (A \cup B) \setminus (A \cap B)$. Similarly, $(A \cup B) \setminus (A \cap B) \subseteq (A \setminus B) \cup (B \setminus A)$. And the equality follows. Observe that $x \in A$ or $x \in B$, but $x \notin$ of both A and B.

   (d) If $(x, y) \in (A \cap B) \times C$, then $x \in A \cap B$ and $y \in C$. Thus, $x \in A$ and $x \in B$. So $(x, y) \in A \times C$, and $(x, y) \in B \times C$. Hence, $(x, y) \in (A \times C) \cap (B \times C)$. And therefore, $(A \cap B) \times C \subseteq (A \times C) \cap (B \times C)$. Similarly, $(A \times C) \cap (B \times C) \subseteq (A \cap B) \times C$. And the equality follows.

   (e) First, observe that $(A \cap B) \cap (A \setminus B) = \phi$. For if $x \in (A \cap B) \cap (A \setminus B)$, then $x \in A \cap B$ and $x \in A \setminus B$. But this means that $x \in A$, and that x is both in B and not in B, which is impossible. Secondly, suppose that $x \in A$. Then, x is either in B or not in B. If $x \in B$, then $x \in A \cap B$. And if $x \notin B$, then $x \in A \setminus B$. Thus, in either case $x \in (A \cap B) \cup (A \setminus B)$, implying that $A \subseteq (A \cap B) \cup (A \setminus B)$. Similarly, $(A \cap B) \cup (A \setminus B) \subseteq A$. And the equality follows.

7. ($\Rightarrow$) Start with a square with sides of length $a+b$. Form four right triangles inside the square with short sides along the original square, one side of length a and the other of length b. Since all four triangles are congruent (i.e., two sides and the included angle are equal), the hypotenuse of each of these triangles is the same, call it c. This creates a small square inside the large square, with sides of length c. Why? Thus, the area inside the large square is equal to the area of four right triangles and a little square. This gives $(a+b)^2 = 4\left(\frac{1}{2}ab\right)+c^2$, which reduces to $a^2+b^2=c^2$.

($\Leftarrow$) Let us consider two triangles. One is a right triangle with short sides (legs) of length a and b. The second triangle has sides a, b, c, for which $a^2+b^2=c^2$. In the first triangle, by proof of ($\Rightarrow$), we have that hypotenuse x satisfies $x^2=a^2+b^2$. Since $a^2+b^2=c^2$, we must have $x^2=c^2$, which implies that $x=c$. Therefore, two triangles have equal sides, hence are congruent. Thus, corresponding angles are equal. It follows that the angle opposite side c in the second triangle must be a right angle.

9. $P(A)$ contains exactly 2^n elements, a *power* of 2.

Section 1.2

1. **(b)** The given equation is equivalent to $\left(x-\frac{1}{2}\right)^2+\left(y+\frac{3}{2}\right)^2=\frac{1}{4}$. To graph this relation start with a circle with the center at the origin and the radius $\frac{1}{2}$ and then shift it to the right $\frac{1}{2}$ units and down $\frac{3}{2}$ units.

2. **(a)** In order for the radical to make sense and to avoid division by 0, we need $x+1>0$. Therefore, $D_f=\{x \mid x>-1\}$.

(b) In order for the radical to make sense, we need $x-1\geq 0$. And to avoid division by 0, we need $x-2\neq 0$. Thus, $D_f=\{x \mid x\geq 1 \text{ and } x\neq 2\}$.

(d) We will avoid division by zero if $x^2+x-2=(x-1)(x+2)\neq 0$. Therefore, $D_f=\{x \mid x\neq 1, x\neq -2\}$.

3. **(a)** To show that f is an injection, suppose that $f(x_1)=f(x_2)$. And then show that $x_1=x_2$ for any $x_1,x_2\in D_f$. Thus, if $2x_1-1=2x_2-1$, we have $2x_1=2x_2$ and so, $x_1=x_2$. To show that f is a surjection, we choose an arbitrary $y=y_1\in R_f$ and find $x\in D_f$ so that $f(x)=y_1$. Now since $2x-1=y_1$, solving for x we obtain $x=\dfrac{y_1+1}{2}\in D_f$. Thus we have $f\left(\dfrac{y_1+1}{2}\right)=y_1$. Hence, f is a bijection.

(b) To show that f is an injection, suppose that $f(x_1)=f(x_2)$. Then $\dfrac{(x_1)^2-1}{x_1-1}=\dfrac{(x_2)^2-1}{x_2-1}$. Therefore, $\dfrac{(x_1-1)(x_1+1)}{x_1-1}=\dfrac{(x_2-1)(x_2+1)}{x_2-1}$. And canceling we obtain $x_1+1=x_2+1$, which yields $x_1=x_2$. Now to show that f is not a surjection, pick a particular value for y, say $y=2$. We will show that there is no value $x\in D_f$ so that $f(x)=2$. Therefore, we solve $\dfrac{x^2-1}{x-1}=2$. Hence, $x^2-2x+1=0$. But the only real solution is $x=1$ which is not in D_f. Thus, f is not a surjection. Note that $y=2$ was chosen since f can be written as $f(x)=\dfrac{x^2-1}{x-1}=x+1$, for $x\neq 1$. Also, $x+1=2$ only if $x=1$ which is not in D_f.

(c) Use steps similar to those in part (a) to show that f is a bijection.

(d) To show that f is not an injection, pick $x_1\neq x_2$. Then $f(x_1)=f(x_2)$. For example, consider $x_1=-1$ and $x_2=1$. When sketching the graph of f, it is apparent that f does not pass the horizontal line test. To show that f is not a surjection, observe that $R_f=\{y \mid 0\leq y\leq 1\}$ is not equal to the interval $[0,4)$.

5. Let $M=\max f$. We will prove that $\sup f=M$. Since $M=\max f$, by Definition 1.2.15, part (b), there exists $x_1 \in D_f$ such that $f(x_1) = M$ and $f(x) \le f(x_1)$ for all $x \in D_f$. Thus, by Definition 1.2.14, M is an upper bound of f. Now, if M_1 is any real number smaller than M, then $M_1 < f(x_1)$, and so M_1 is not an upper bound of f. Hence, M is the least upper bound of f meaning that $M = \sup f$.

7. **(b)** Since f, g are odd functions, $f(-x) = -f(x)$ and $g(-x) = -g(x)$ for all $x \in A$. Thus, for all $x \in A$, we have $(fg)(-x) = f(-x)g(-x) = [-f(x)][-g(x)] = f(x)g(x) = (fg)(x)$, and so fg is even.

(c) $f(x) = 0$ for all $x \in \Re$.

9. **(a)** $\frac{1}{2}(f+g)(x) + \frac{1}{2}|(f-g)(x)| = \frac{1}{2}f(x) + \frac{1}{2}g(x) + \left|\frac{1}{2}f(x) - \frac{1}{2}g(x)\right| = \frac{1}{2}f(x) + \frac{1}{2}g(x) +$
$$\begin{cases} \frac{1}{2}f(x) - \frac{1}{2}g(x) & \text{if } f(x) \ge g(x) \\ -\left[\frac{1}{2}f(x) - \frac{1}{2}g(x)\right] & \text{if } f(x) < g(x) \end{cases} = \begin{cases} f(x) & \text{if } f(x) \ge g(x) \\ g(x) & \text{if } f(x) < g(x) \end{cases} = (f \vee g)(x).$$

(c) $(f \vee g)(x) = x^2$ if $x < -1$ or $x > 1$, $(f \vee g)(x) = 2 - x^2$ if $-1 \le x \le 1$, and $(f \wedge g)(x) = 2 - x^2$ if $x < -1$ or $x > 1$, $(f \wedge g)(x) = x^2$ if $-1 \le x \le 1$.

10. $(f \circ g)(-2) = f(g(-2)) = f(2(-2)+1) = f(-4+1) = f(-3) = (-3)^2 - 3 = 6$

11. Choose $f(x) = x^2$ and $g(x) = x + 1$. Then, $(f \circ g)(x) = x^2 + 2x + 1$ and $(g \circ f)(x) = x^2 + 1$.

12. Since $(g \circ f)(-x) = g(f(-x)) = g(-f(x)) = g(f(x)) = (g \circ f)(x)$, $g \circ f$ is an even function.
Since $(g \circ g)(-x) = g(g(-x)) = g(g(x)) = (g \circ g)(x)$, $g \circ g$ is an even function.

13. **(a)** Suppose $f : A \to B$ and $g : B \to C$ are injective. To prove $g \circ f : A \to C$ is also injective, we choose $x_1, x_2 \in A$ such that $(g \circ f)(x_1) = (g \circ f)(x_2)$. We need to show that $x_1 = x_2$. To this end, note that $(g \circ f)(x_1) = (g \circ f)(x_2)$ implies that $g(f(x_1)) = g(f(x_2))$, and since g is injective we have $f(x_1) = f(x_2)$. But, in turn, f is injective, which implies that $x_1 = x_2$, which is what we needed to prove.

(b) Suppose $f : A \to B$ and $g : B \to C$ are surjective. To prove $g \circ f : A \to C$ is also surjective, we choose any $c \in C$ and show that there exists $a \in A$ such that $(g \circ f)(a) = c$. To this end, since f and g are surjective, there exists $b \in B$ such that $g(b) = c$ and $a \in A$ such that $f(a) = b$. Thus, $(g \circ f)(a) = g(f(a)) = g(b) = c$.

14. The function f is symmetric with respect to the point (a, b) if and only if whenever $(a - x, b - y)$ is on the graph(f), then so is $(a + x, b + y)$. If in addition, $a \in D_f$, then we can write that f is symmetric with respect to the point (a, b) if and only if $f(a - x) + f(a + x) = 2f(a)$ for all $x \in D_f$.

15. Any function on $\Re$, on $[-L, L]$, or on $(-L, L)$ can be written as $f(x) = \dfrac{f(x) + f(-x)}{2} + \dfrac{f(x) - f(-x)}{2}$. Now verify that the last two quotients are even and odd, respectively.

16. Since $f(n+1) = f(n) + \dfrac{1}{3}$, use patterns to show that $f(n) = \dfrac{1}{3}(n-1)$. Thus, $f(79) = 26$.

18. Proofs follow from the definitions.

19. **(a)** ($\Rightarrow$) Assume that $y \in f(A \cup B)$. We will show that $y \in f(A) \cup f(B)$. Since $y \in f(A \cup B)$, there exists $x \in A \cup B$ such that $f(x) = y$. Therefore, $x \in A$ or $x \in B$. If $x \in A$, then $y = f(x) \in f(A)$. Similarly, if $x \in B$, then $y = f(x) \in f(B)$. So, $y \in f(A) \cup f(B)$.

($\Leftarrow$) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, by Exercise 18(a), we have that $f(A) \subseteq f(A \cup B)$ and $f(B) \subseteq f(A \cup B)$. Hence, $f(A) \cup f(B) \subseteq f(A \cup B)$.

(b) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, by Exercise 18(a), we have that $f(A \cap B) \subseteq f(A)$ and $f(A \cap B) \subseteq f(B)$. Hence, the desired conclusion follows.

(c) Assume $y \in f(A) \setminus f(B)$. Then $y \in f(A)$ and $y \notin f(B)$. Hence, there exists $x \in A$ such that $f(x) = y$. Since $y \notin f(B)$, we know that $x \notin B$. Therefore, $x \in A \setminus B$ and $y = f(x) \in f(A \setminus B)$. Thus, the desired conclusion follows.

20. (a) ($\Rightarrow$) We will show that $f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B)$. Let $x \in f^{-1}(A \cup B)$, then $f(x) \in A \cup B$. So $f(x) \in A$ or $f(x) \in B$. If $f(x) \in A$, then $x \in f^{-1}(A)$. Similarly, if $f(x) \in B$, then $x \in f^{-1}(B)$. Thus, $x \in f^{-1}(A) \cup f^{-1}(B)$ and the desired conclusion follows.

($\Leftarrow$) Let $x \in f^{-1}(A) \cup f^{-1}(B)$. Then, $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$. If $x \in f^{-1}(A)$, then $f(x) \in A$. If $x \in f^{-1}(B)$, then $f(x) \in B$. Thus, $f(x) \in A \cup B$ which implies that $x \in f^{-1}(A \cup B)$. It follows that $f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B)$.

(b) ($\Rightarrow$) For contrast, we will use a different style of proof than in Part (a). Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, by Exercise 18(b), we have that $f^{-1}(A \cap B) \subseteq f^{-1}(A)$ and $f^{-1}(A \cap B) \subseteq f^{-1}(B)$. Hence, $f^{-1}(A \cap B) \subseteq f^{-1}(A) \cap f^{-1}(B)$.

($\Leftarrow$) Let $x \in f^{-1}(A) \cap f^{-1}(B)$. Then, $x \in f^{-1}(A)$ and $x \in f^{-1}(B)$. Since $x \in f^{-1}(A)$ implies that $f(x) \in A$, and $x \in f^{-1}(B)$ implies that $f(x) \in B$, we have $f(x) \in A \cap B$. Thus $x \in f^{-1}(A \cup B)$, which proves that $f^{-1}(A) \cap f^{-1}(B) \subseteq f^{-1}(A \cap B)$.

(c) ($\Rightarrow$) Let $x \in f^{-1}(A \setminus B)$, then $f(x) \in A \setminus B$. So $f(x) \in A$ and $f(x) \notin B$. Thus, $x \in f^{-1}(A)$ and $x \notin f^{-1}(B)$. Therefore, $x \in f^{-1}(A) \setminus f^{-1}(B)$. Hence, $f^{-1}(A \setminus B) \subseteq f^{-1}(A) \setminus f^{-1}(B)$.

($\Leftarrow$) Let $x \in f^{-1}(A) \setminus f^{-1}(B)$. Then, $x \in f^{-1}(A)$ and $x \notin f^{-1}(B)$. So $f(x) \in A$ and $f(x) \notin B$. Thus, $f(x) \in A \setminus B$. So $x \in f^{-1}(A \setminus B)$. It follows that $f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B)$.

21. (a) Let $x \in A$. Then $f(x) \in f(A)$. This implies that $x \in f^{-1}(f(A))$. Hence, the conclusion follows.

(b) Let $y \in f(f^{-1}(B))$. Then there exists $x \in f^{-1}(B)$ such that $y = f(x)$. Since $x \in f^{-1}(B)$, we have $y = f(x) \in B$. Hence, the conclusion follows.

22. (a) In Exercise 19(b) we proved that $f(A \cap B) \subseteq f(A) \cap f(B)$ for any sets A and B in X. To prove the reverse inclusion we will need the assumption that f is injective. To this end, let $y \in f(A) \cap f(B)$. Then $y \in f(A)$ and $y \in f(B)$. Since $y \in f(A)$, there exists $a \in A$ such that $f(a) = y$. Similarly, since $y \in f(B)$, there exists $b \in B$ such that $f(b) = y$. Therefore, $f(a) = f(b)$. Since f is injective, $a = b$. Thus, $a \in A$ and $a \in B$. So, $a \in A \cap B$. Hence, $y = f(a) \in f(A \cap B)$, and so, $f(A) \cap f(B) \subseteq f(A \cap B)$.

(c) In view of Exercise 21(a), we only need to show that $f^{-1}(f(A)) \subseteq A$ if f is injective. Thus, let $x \in f^{-1}(f(A))$. Then $f(x) \in f(A)$. So, there exists $a \in A$ such that $f(x) = f(a)$. Since f is injective, $x = a$ and so, $x \in A$. Hence, $f^{-1}(f(A)) \subseteq A$.

23. (a) By Exercise 21(a) we have $A \subseteq f^{-1}(f(A))$ for any $A \subseteq X$. By Exercise 22(c) we have $f^{-1}(f(A)) \subseteq A$ only if f is injective. Thus the desired equality might not hold only if f is not injective. Let us choose $f(x) = x^2$, $A = [0,1]$, and $X = \mathfrak{R}$. Then $f^{-1}(f(A)) = [-1,1]$ which is not in A.

24. (a) In view of Exercises 19(b) and 22(a), we are looking for a function f which is not injective and that $f(A) \cap f(B)$ is not a subset of $f(A \cap B)$. Let us choose $f(x) = x^2$, $A = (-\infty,0]$, and $B = [0,\infty)$. Then, $f(A) = f(B) = [0,\infty)$. Since $A \cap B = \{0\}$ we have $f(A \cap B) = \{0\}$. Obviously, $f(A) \cap f(B) \neq f(A \cap B)$.

(b) Let $f(x) = x^2$, $A = (-\infty,0]$, and $B = [0,\infty)$. Then, $A \setminus B = (-\infty,0)$, $f(A \setminus B) = (0,\infty)$, $f(A) = [0,\infty) = f(B)$, and $f(A) \setminus f(B) = \phi$. Obviously, $\phi \neq (0,\infty)$.

Section 1.3

1. **(b)** Suppose n and m are odd integers. Then there exist $r,s \in Z$ such that $n = 2r+1$ and $m = 2s+1$. So, $nm = (2r+1)(2s+1) = 2(2rs+r+s)+1$, which is an odd integer.

2. **(a)** Suppose $P(n)$ is the statement "$n^2 + n$ is divisible by 2." Then $P(1)$ is true because $1^2 + 1 = 2$ is divisible by 2. Next, suppose $P(k)$ is true for some $k \in N$, that is, $k^2 + k$ is divisible by 2. We will show that $P(k+1)$ is true, that is, $(k+1)^2 + (k+1)$ is divisible by 2. To this end, we write $(k+1)^2 + (k+1) = k^2 + 2k + 1 + k + 1 = \left(k^2 + k\right) + 2(k+1)$. Since $k^2 + k$ and $2(k+1)$ are divisible by 2, then so is the sum. Hence, $P(n)$ is true for all $n \in N$.

(b) Suppose $P(n)$ is the statement "$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$." Then $P(1)$ is true. Next, suppose $P(k)$ is true for some $k \in N$, that is, $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$. We will show that $P(k+1)$ is true, that is, $\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$. To this end, we write $\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6}$ $+ (k+1)^2 = \frac{k+1}{6}(k+2)(2k+3)$. Hence, $P(n)$ is true for all $n \in N$.

(c) Suppose $P(n)$ is the statement "$\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$." Then $P(1)$ is true. Next, suppose $P(k)$ is true for some $k \in N$, that is, $\sum_{i=1}^{k} i^3 = \left[\frac{k(k+1)}{2}\right]^2$. We will show that $P(k+1)$ is true, that is, $\sum_{i=1}^{k+1} i^3 =$ $\left[\frac{(k+1)(k+2)}{2}\right]^2$. To this end, write $\sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} i^3 + (k+1)^3 = \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3 = \frac{(k+1)^2}{4}(k+2)^2$. Hence, $P(n)$ is true for all $n \in N$.

(d) Suppose $P(n)$ is the statement "$\sum_{k=1}^{n} (2k-1) = n^2$." Then $P(1)$ is true. Next, suppose $P(k)$ is true for some $k \in N$, that is, $\sum_{i=1}^{k} (2i-1) = k^2$. We will show that $P(k+1)$ is true, that is, $\sum_{i=1}^{k+1} (2i-1) =$ $(k+1)^2$. To this end, we write $\sum_{i=1}^{k+1} (2i-1) = \sum_{i=1}^{k} (2i-1) + [2(k+1)-1] = k^2 + 2k + 2 - 1 = (k+1)^2$. Hence, $P(n)$ is true for all $n \in N$.

(f) Suppose $P(n)$ is the statement "$0 < x^n < 1$." Then $P(1)$ is true. Next, suppose $P(k)$ is true for some $k \in N$, that is, $0 < x^k < 1$. We will show that $P(k+1)$ is true, that is, $0 < x^{k+1} < 1$. To this end, we write $0 < x^{k+1} = x^k(x) < 1(x) < 1$. Hence, $P(n)$ is true for all $n \in N$.

(g) Suppose $P(n)$ is the statement "$2^{n-1} \le n! \le n^n$." Then $P(1)$ says that $2^0 \le 1! \le 1$, which is true. Next, suppose $P(k)$ is true for some $k \in N$, that is, $2^{k-1} \le k! \le k^k$. We will show that $P(k+1)$ is true, that is, $2^k \le (k+1)! \le (k+1)^{k+1}$. To this end, we write $2^k = 2^{k-1}(2) \le (k!)(2) \le (k!)(k+1) \le$ $k^k(k+1) \le (k+1)^k(k+1) = (k+1)^{k+1}$. Hence, $P(n)$ is true for all $n \in N$.

(i) Suppose $P(n)$ is the statement "$\cos n\pi = (-1)^n$." Then $P(1)$ is true. Next, suppose $P(k)$ is true for some $k \in N$, that is, $\cos k\pi = (-1)^k$. We will show that $P(k+1)$ is true, that is, $\cos(k+1)\pi =$ $(-1)^{k+1}$. To this end, we write $\cos(k+1)\pi = \cos(k\pi + \pi) = \cos k\pi \cos \pi - \sin k\pi \sin \pi =$ $(\cos k\pi)(-1) - 0 = (-1)^k(-1) = (-1)^{k+1}$. Hence, $P(n)$ is true for all $n \in N$.

(j) Suppose $P(n)$ is the statement "$\frac{n^5}{5}+\frac{n^3}{3}+\frac{7n}{15}$ is an integer for every $n \in N$." Then $P(1)$ is true. Next, suppose $P(k)$ is true for some $k \in N$, that is, $\frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}$ is an integer. We will show that $P(k+1)$ is true, that is, $\frac{(k+1)^5}{5}+\frac{(k+1)^3}{3}+\frac{7(k+1)}{15}$ is an integer. To this end, using Pascal's triangle, we write $\frac{(k+1)^5}{5}+\frac{(k+1)^3}{3}+\frac{7(k+1)}{15}=\frac{k^5+5k^4+10k^3+10k^2+5k+1}{5}+\frac{k^3+3k^2+3k+1}{3}+\frac{7k+7}{15}=$ $\frac{k^5}{5}+\left(k^4+2k^3+2k^2+k\right)+\frac{1}{5}+\frac{k^3}{3}+\left(k^2+k\right)+\frac{1}{3}+\frac{7k}{15}+\frac{7}{15}=\left(\frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}\right)+$ $\left(k^4+2k^3+3k^2+2k+1\right)$. This is an integer since the sum of integers is an integer. Hence, $P(n)$ is true for all $n \in N$.

(k) Suppose $P(n)$ is the statement "$(n-1)(n)(n+1)$ is divisible by 6 for every $n \in N$." Then $P(1)$ is true. Next, suppose $P(k)$ is true for some $k \in N$, that is, $(k-1)(k)(k+1)$ is divisible by 6. We will show that $P(k+1)$ is true, that is, $k(k+1)(k+2)$ is divisible by 6. To this end, we write $k(k+1)(k+2)$ $=k^3+3k^2+2k=\left(k^3-k\right)+\left(3k^2+3k\right)=(k-1)(k)(k+1)+3\left(k^2+k\right)$. Since $(k-1)(k)(k+1)$ is divisible by 6; and by Exercise 2(a), k^2+k is divisible by 2, $k(k+1)(k+2)$ is a sum of two integers divisible by 6 and so, it is divisible by 6. Hence, $P(n)$ is true for all $n \in N$.

(l) Suppose $P(n)$ is the statement "n^5-n is divisible by 5 for every $n \in N$." Then $P(1)$ is true. Next, suppose $P(k)$ is true for some $k \in N$, that is, k^5-k is divisible by 5. We will show that $P(k+1)$ is true, that is, $(k+1)^5-(k+1)$ is divisible by 5. To this end, we write $(k+1)^5-(k+1)=$ $k^5+5k^4+10k^3+10k^2+5k+1-k-1=\left(k^5-k\right)+5\left(k^4+2k^3+2k^2+k\right)$, which is a sum of two integers, each divisible by 5. Thus, $(k+1)^5-(k+1)$ is also divisible by 5. Hence, $P(n)$ is true for all $n \in N$.

(m) Suppose $P(n)$ is the statement "$2^{2n+1}+1$ is divisible by 3 for every $n \in N$." Then $P(1)$ is true. Next, suppose $P(k)$ is true for some $k \in N$, that is, $2^{2k+1}+1$ is divisible by 3. We will show that $P(k+1)$ is true, that is, $2^{2k+3}+1$ is divisible by 3. To this end, we can write $2^{2k+3}+1=2^{2k+1}(2^2)+1=$ $4\left(2^{2k+1}+1\right)-3$, which is divisible by 3 since both terms are. Hence, $P(n)$ is true for all $n \in N$.

(n) Suppose $P(n)$ is the statement "$a \le \frac{x_1+x_2+\cdots+x_n}{n} \le b$." Then $P(1)$ is true since $a \le x_1 \le b$. Next, suppose for some $k \in N$, $a \le \frac{x_1+x_2+\cdots+x_k}{k} \le b$. We will show that $a \le \frac{x_1+x_2+\cdots+x_k+x_{k+1}}{k+1}$ $\le b$. Since induction hypotheses imply that $ka \le x_1+x_2+\cdots+x_k \le kb$, and since $a \le x_{k+1} \le b$, addition gives $ka+a \le x_1+x_2+\cdots+x_k+x_{k+1} \le kb+b$. Division by $k+1$ yields the desired result. Hence, $P(n)$ is true for all $n \in N$.

(o) Suppose $P(n)$ is the statement "$\frac{n}{n+1} \ge \frac{1}{2}$ for all $n \in N$." Then $P(1)$ is true. Next, suppose $P(k)$ is true for some $k \in N$, that is, $\frac{k}{k+1} \ge \frac{1}{2}$. We will show that $P(k+1)$ is true, that is, $\frac{k+1}{k+2} \ge \frac{1}{2}$. Since $k^2+2k+1 \ge k^2+2k$, we have that $(k+1)(k+1) \ge k(k+2)$. Since each term is positive, this gives $\frac{k+1}{k+2} \ge \frac{k}{k+1}$. Thus, $\frac{k+1}{k+2} \ge \frac{1}{2}$. Hence, $P(n)$ is true for all $n \in N$.

(p) Suppose $P(n)$ is the statement "$0<\frac{2^n}{n!} \le 2\left(\frac{2}{3}\right)^{n-2}$ for all natural numbers $n \ge 3$." Then $P(3)$ is true.

Next, suppose $P(k)$ is true for some integer $k \geq 3$, that is, $0 < \frac{2^k}{k!} \leq 2\left(\frac{2}{3}\right)^{k-2}$. We will show that $P(k+1)$ is true, that is, $0 < \frac{2^{k+1}}{(k+1)!} \leq 2\left(\frac{2}{3}\right)^{k-1}$. To this end, we write $0 < \frac{2^{k+1}}{(k+1)!} = \frac{2^k 2}{(k!)(k+1)} \leq 2\left(\frac{2}{3}\right)^{k-2}\left(\frac{2}{k+1}\right) \leq 2\left(\frac{2}{3}\right)^{k-2}\left(\frac{2}{3}\right) = 2\left(\frac{2}{3}\right)^{k-1}$. Hence, $P(n)$ is true for all $n \geq 3$.

(q) Suppose $P(n)$ is the statement "$\sum_{k=1}^{n} k(k!) = (n+1)! - 1$ for all $n \in N$." Then $P(1)$ is true. Next, suppose $P(k)$ is true for some $k \in N$, that is, $\sum_{i=1}^{k} i(i!) = (k+1)! - 1$. We will show that $P(k+1)$ is true, that is, $\sum_{i=1}^{k+1} i(i!) = (k+2)! - 1$. To this end, we write $\sum_{i=1}^{k+1} i(i!) = \sum_{i=1}^{k} i(i!) + (k+1)(k+1)! =$ $[(k+1)! - 1] + (k+1)(k+1)! = [(k+1)![1 + (k+1)] - 1 = (k+2)! - 1$. So, $P(n)$ is true for all $n \in N$.

(s) Suppose $P(n)$ is the statement "$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} > \sqrt{n}$ for all natural numbers $n \geq 2$." Then $P(2)$ is true. Next, suppose $P(k)$ is true for some $k \geq 2$, that is, $\sum_{i=1}^{k} \frac{1}{\sqrt{i}} > \sqrt{k}$. We show that $P(k+1)$ is true, that is, $\sum_{i=1}^{k+1} \frac{1}{\sqrt{i}} > \sqrt{k+1}$. To this end, we write $\sum_{i=1}^{k+1} \frac{1}{\sqrt{i}} = \sum_{i=1}^{k} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{k+1}} > \sqrt{k} + \frac{1}{\sqrt{k+1}}$. We need to show that $\sqrt{k} + \frac{1}{\sqrt{k+1}} \geq \sqrt{k+1}$. In order to do this, note that $\sqrt{k+1} > \sqrt{k}$ for $k \geq 2$. Now multiply both sides by $\sqrt{k}$ to get $\sqrt{k}\sqrt{k+1} > k$, which gives $\sqrt{k}\sqrt{k+1} + 1 > k+1$. Division by $\sqrt{k+1}$ yields the desired expression. Hence, $P(n)$ is true for all $n \geq 2$.

(t) Suppose $P(n)$ is the statement "$\sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n}$ for all $n \in N$." Then $P(1)$ is true. Next, suppose $P(k)$ is true for some $k \in N$, that is, $\sum_{i=1}^{k} \frac{1}{i^2} \leq 2 - \frac{1}{k}$. We will show that $P(k+1)$ is true, that is, $\sum_{i=1}^{k+1} \frac{1}{i^2} \leq 2 - \frac{1}{k+1}$. To this end, we can write $\sum_{i=1}^{k+1} \frac{1}{i^2} = \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k+1}$, because $\frac{1}{k} - \frac{1}{(k+1)^2} \geq \frac{1}{k+1}$, which is true because $k^2 + k + 1 \geq k^2 + k$. Hence, $P(n)$ is true for all $n \in N$.

(u) Suppose $P(n)$ is the statement "$\sum_{k=1}^{n} \frac{1}{k^2} \leq \frac{7}{4} - \frac{1}{n}$ for all natural numbers $n \geq 2$." Then $P(2)$ is true. Next, suppose $P(k)$ is true for some integer $k \geq 2$, that is, $\sum_{i=1}^{k} \frac{1}{i^2} \leq \frac{7}{4} - \frac{1}{k}$. We will show that $P(k+1)$ is true, that is, $\sum_{i=1}^{k+1} \frac{1}{i^2} \leq \frac{7}{4} - \frac{1}{k+1}$. To this end, we write $\sum_{i=1}^{k+1} \frac{1}{i^2} = \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2} \leq \frac{7}{4} - \frac{1}{k} + \frac{1}{(k+1)^2} \leq \frac{7}{4} - \frac{1}{k+1}$ because $\frac{1}{k} - \frac{1}{(k+1)^2} \geq \frac{1}{k+1}$, which in turn is true because $k^2 + k + 1 \geq k^2 + k$. Hence, $P(n)$ is true for all $n \geq 2$.

(v) Suppose $P(n)$ is the statement "$2\sum_{k=1}^{n} \frac{1}{k^3} < 3 - \frac{1}{n^2}$ for all natural numbers $n \geq 2$." Then $P(2)$ is true.

Suppose $P(k)$ is true for some integer $k \geq 2$, that is, $2\sum_{i=1}^{k}\frac{1}{i^3} < 3-\frac{1}{k^2}$. We will show that $P(k+1)$ is true, that is, $2\sum_{i=1}^{k+1}\frac{1}{i^3} < 3-\frac{1}{(k+1)^2}$. To this end, we write $2\sum_{i=1}^{k+1}\frac{1}{i^3} = 2\sum_{i=1}^{k+1}\frac{1}{i^3}+\frac{2}{(k+1)^3} < 3-\frac{1}{k^2}+\frac{2}{(k+1)^3}$ $< 3-\frac{1}{(k+1)^2}$, which is true because $k^3+k^2+3k+1 \geq k^3+k^2$. Hence, $P(n)$ is true for all $n \geq 2$.

(w) Suppose $P(n)$ is the statement "$\sum_{k=1}^{n}\frac{1}{(2k-1)(2k+1)} = \frac{n}{2n+1}$ for all $n \in N$." Then $P(1)$ is true. Next, suppose $P(k)$ is true for some $k \in N$, that is, $\sum_{i=1}^{k}\frac{1}{(2i-1)(2i+1)} = \frac{k}{2k+1}$. We will show that $P(k+1)$ is true, that is, $\sum_{i=1}^{k+1}\frac{1}{(2i-1)(2i+1)} = \frac{k+1}{2k+3}$. Thus we have that $\sum_{i=1}^{k+1}\frac{1}{(2i-1)(2i+1)} = \sum_{i=1}^{k}\frac{1}{(2i-1)(2i+1)}+$ $\frac{1}{[2(k+1)-1][2(k+1)+1]} = \frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)} = \frac{k+1}{2k+3}$. Hence, $P(n)$ is true for all $n \in N$.

3. **(a)** From Example 1.3.4 we have $1+x+x^2+\cdots+x^n = \frac{x^{n+1}-1}{x-1}$, if $x \neq 1$. Therefore, $-x-x^2-\cdots-x^n$ $=1-\frac{x^{n+1}-1}{x-1}$. Hence, $1-x-x^2-\cdots-x^n = 2-\frac{x^{n+1}-1}{x-1} = \frac{x^{n+1}-2x+1}{1-x} = 2-\frac{1-x^{n+1}}{1-x}$, if $x \neq 1$. If $x=1$, then the desired sum has the value $1-n$.

(b) $\sum_{k=20}^{30}k^2 = \sum_{k=1}^{30}k^2 - \sum_{k=1}^{19}k^2 = 9{,}455-2{,}470 = 6{,}985.$

4. **(a)** 15

(b) $\binom{n}{0} = \frac{n!}{(0!)(n-0)!} = 1$ and $\binom{n}{n} = \frac{n!}{(n!)(n-n)!} = 1.$

(d) $\binom{n}{k} = \frac{n!}{(k!)(n-k)!}$ and thus, $\binom{n}{n-k} = \frac{n!}{(n-k)![n-(n-k)]!} = \frac{n!}{(n-k)!(k!)} = \binom{n}{k}$.

(e) $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}(1^{n-k}) = (1-1)^n = 0^n = 0$.

(f) From the binomial theorem, where $a=2$, $b=-x$, and $n=7$, the term with x^3 is given by $\binom{7}{3}2^4(-x)^3 = -560x^3$.

(g) $2^n = (1+1)^n = 1+n+\frac{n(n-1)}{2}+\frac{n(n-1)(n-2)}{6}+\cdots+1 > \frac{n(n-1)(n-2)}{6}$, since we dropped only nonnegative terms.

(h) $(1+x)^{n+1} = \binom{n+1}{0}+\binom{n+1}{1}x+\cdots+\binom{n+1}{k}x^k+\cdots+\binom{n+1}{n+1}x^{n+1}$, and $(1+x)^n(1+x) =$ $\left[\binom{n}{0}+\binom{n}{1}x+\cdots+\binom{n}{n}x^n\right](1+x) = \binom{n}{0}+\left[\binom{n}{0}+\binom{n}{1}\right]x+\cdots+\left[\binom{n}{k-1}+\binom{n}{k}\right]x^k+\cdots+$ $\left[\binom{n}{n-1}+\binom{n}{n}\right]x^n+\binom{n}{n}x^{n+1}$. Now, since $(1+x)^{n+1} = (1+x)^n(1+x)$, equate coefficients to obtain the desired relation.

(i) $2^1 + 2^2 + \cdots + 2^{50} = 2\left(1 + 2^1 + 2^2 + \cdots + 2^{49}\right) = 2\left(\dfrac{1-2^{50}}{1-2}\right) = 2\left(2^{50} - 1\right).$

(j) $\dbinom{n}{k} = \dfrac{n(n-1)(n-2)\cdot\cdots\cdot(n-(k-1))}{k!} \le \dfrac{n(n)(n)\cdot\cdots\cdot(n)}{k!} = \dfrac{n^k}{k!}$

5. **(b)** $(1+x)^n = 1 + nx + \frac{1}{2}n(n-1)x^2 + \cdots + x^n \ge 1 + nx$, since all other terms are ≥ 0.

6. **(a)** Suppose $P(n)$ is the statement "$a_n < \left(\frac{7}{4}\right)^n$" and use Theorem 1.3.9. Clearly, $P(1)$ and $P(2)$ are true. Next, let k be any integer such that $k > 2$ and suppose $P(i)$ is true for all $i = 1, 2, \ldots, k$, that is, $a_i < \left(\frac{7}{4}\right)^i$ for positive integer $i \le k$. We need to show that $P(k+1)$ is true, that is, $a_{k+1} < \left(\frac{7}{4}\right)^{k+1}$. To this end, we write $a_{k+1} = a_k + a_{k-1} < \left(\frac{7}{4}\right)^k + \left(\frac{7}{4}\right)^{k-1} = \left(\frac{7}{4}\right)^{k-1}\left[\frac{7}{4} + 1\right] < \left(\frac{7}{4}\right)^{k-1}\left(\frac{7}{4}\right)^2 = \left(\frac{7}{4}\right)^{k+1}$. Hence, $P(n)$ is true for all $n \in N$.

(c) Suppose $P(n)$ is the statement "$a_n < a_{n+1}$." Then $P(1)$ is true. Next, suppose $P(k)$ is true for some $k \in N$, that is, $a_k < a_{k+1}$. We will show that $P(k+1)$ is true, that is, $a_{k+1} < a_{k+2}$. To this end, we write $a_{k+1} = \sqrt{3a_k + 1} < \sqrt{3a_{k+1} + 1} = a_{k+2}$. Hence, $P(n)$ is true for all $n \in N$.

7. **(h)** Method of Example 1.3.12 does not require that the explicit formula be found by trial and error and then proven by induction. Set $a_n = cr^n$ in the recursion formula $a_{n+1} = \frac{5}{6}a_n - \frac{1}{6}a_{n-1}$, and rewrite it to obtain $cr^{n+1} = \frac{5}{6}cr^n - \frac{1}{6}cr^{n-1}$. Next, since $c \ne 0$, divide both sides by cr^{n-1} to obtain $r^2 = \frac{5}{6}r - \frac{1}{6}$. Solutions of this equation are $r = \frac{1}{2}$ and $r = \frac{1}{3}$. Thus, $a_n = \frac{c_1}{2^n}$ or $a_n = \frac{c_2}{3^n}$, where the previous c is now denoted by c_1 or c_2. In fact, a_n can be written as $a_n = \frac{c_1}{2^n} + \frac{c_2}{3^n}$. Next, we need to find c_1 and c_2. By knowing the first two terms of the sequence, we find that $c_1 = 0$ and $c_2 = 3$. Thus, $a_n = \frac{0}{2^n} + \frac{3}{3^n} = \frac{1}{3^{n-1}}$.

8. In order to prove Jensen's inequality, we will prove that if $P(n)$ is the statement "$f\left(\dfrac{x_1 + x_2 + \cdots + x_n}{n}\right) \le \dfrac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}$," then statements (a) and (b) in Theorem 1.3.13 are satisfied. *Step 1.* We will use Theorem 1.3.2 to prove that $P\left(2^m\right)$ is true for every $m \in N$. So, if $m = 1$, we see that $P(2)$ says that $f\left(\dfrac{x_1 + x_2}{2}\right) \le \dfrac{f(x_1) + f(x_2)}{2}$, which is true since f is convex. Next we suppose $P\left(2^k\right)$ is true for some positive integer k, that is, $f\left(\dfrac{x_1 + x_2 + \cdots + x_{2^k}}{2^k}\right) \le \dfrac{f(x_1) + f(x_2) + \cdots + f\left(x_{2^k}\right)}{2^k}$, and we show that $P\left(2^{k+1}\right)$ is true, that is, $f\left(\dfrac{x_1 + x_2 + \cdots + x_{2^k} + x_{2^k+1} + \cdots + x_{2^{k+1}}}{2^{k+1}}\right) \le \dfrac{f(x_1) + f(x_2) + \cdots + f\left(x_{2^{k+1}}\right)}{2^{k+1}}$. To this end, we write $f\left(\frac{x_1+x_2+\cdots+x_{2^k}+x_{2^k+1}+\cdots+x_{2^{k+1}}}{2^{k+1}}\right)=f\left(\frac{\frac{x_1+x_2+\cdots+x_{2^k}}{2^k}}{2}+\frac{\frac{x_{2^k+1}+\cdots+x_{2^{k+1}}}{2^k}}{2}\right)=f\left(\frac{s}{2}+\frac{t}{2}\right)$, (where $s,t\in[a,b]$, by Exercise 2(n)), $\le\frac{f(s)+f(t)}{2}\le\frac{1}{2}\left[f\left(\frac{x_1+x_2+\cdots+x_{2^k}}{2^k}\right)+f\left(\frac{x_{2^k+1}+\cdots+x_{2^{k+1}}}{2^k}\right)\right]\le$ $\frac{1}{2}\left[\frac{f(x_1)+f(x_2)+\cdots+f(x_{2^k})}{2^k}+\frac{f(x_{2^k+1})+\cdots+f(x_{2^{k+1}})}{2^k}\right]=\frac{f(x_1)+f(x_2)+\cdots+f(x_{2^{k+1}})}{2^{k+1}}$, which is what we wanted to verify. Hence, part (a) of Theorem 1.3.13 is satisfied.

Step 2. We will prove that part (b) of Theorem 1.3.13 is satisfied. Thus, suppose $P(k)$ is true for some $k\in N$ and show $P(k-1)$ is also true. Therefore, since $\frac{x_1+x_2+\cdots+x_{k-1}}{k-1}\in[a,b]$, we can write that $f\left(\frac{x_1+x_2+\cdots+x_{k-1}+\frac{x_1+x_2+\cdots+x_{k-1}}{k-1}}{k}\right)\le\frac{f(x_1)+f(x_2)+\cdots+f(x_{k-1})+f\left(\frac{x_1+x_2+\cdots+x_{k-1}}{k-1}\right)}{k}$. Now, observe that we can write $f\left(\frac{x_1+x_2+\cdots+x_{k-1}+\frac{x_1+x_2+\cdots+x_{k-1}}{k-1}}{k}\right)=f\left(\frac{x_1+x_2+\cdots+x_{k-1}}{k-1}\right)$, thus, the previous inequality can be written as $f\left(\frac{x_1+x_2+\cdots+x_{k-1}}{k-1}\right)\le\frac{f(x_1)+f(x_2)+\cdots+f(x_{k-1})}{k}+\frac{1}{k}f\left(\frac{x_1+x_2+\cdots+x_{k-1}}{k-1}\right)$. This can be written as, $\frac{k-1}{k}f\left(\frac{x_1+x_2+\cdots+x_{k-1}}{k-1}\right)\le\frac{f(x_1)+f(x_2)+\cdots+f(x_{k-1})}{k}$, which gives $f\left(\frac{x_1+x_2+\cdots+x_{k-1}}{k-1}\right)\le\frac{f(x_1)+f(x_2)+\cdots+f(x_{k-1})}{k-1}$. Therefore, part (b) is proven. Hence, by Theorem 1.3.13, $P(n)$ is true for all $n\in N$.

9. *Procedure 1.* Prove this result using Theorem 1.3.13, that is, use steps similar to the ones we used in Exercise 8.

Procedure 2. Use Exercise 8 with $f(x)=-\ln x$. To prove f is convex we need Theorem 1.8.4, part (c), and the fact that $\ln x$ is an increasing function.

10. **(a)** The desired sum produces $(n+1)+(n+1)+\cdots+(n+1)=2S$. Thus, $n(n+1)=2S$, and so, $S=\frac{n(n+1)}{2}$.

(b) Since $(k+1)^2-k^2=2k+1$, we have that $\sum_{k=1}^{n}\left[(k+1)^2-k^2\right]=\sum_{k=1}^{n}(2k+1)$. Now, expanding the left-hand side due to the telescoping nature of the sum we get that $\sum_{k=1}^{n}\left[(k+1)^2-k^2\right]=\left(2^2-1^2\right)+\left(3^2-2^2\right)+\ldots+\left[(n+1)^2-n^2\right]=(n+1)^2-1=n^2+2n$. But $\sum_{k=1}^{n}(2k+1)=2\sum_{k=1}^{n}k+n$. Thus, equating the above we get $n^2+2n=2\sum_{k=1}^{n}k+n$. Solving we obtain that $\sum_{k=1}^{n}k=\frac{n^2+n}{2}=\frac{n(n+1)}{2}$.

(c) Observe that $(k+1)^3-k^3=3k^2+3k+1$. Thus, expend both sides of $\sum_{k=1}^{n}\left[(k+1)^3-k^3\right]=\sum_{k=1}^{n}\left(3k^2+3k+1\right)$ and use the fact that $\sum_{k=1}^{n}k=\frac{n(n+1)}{2}$ in order to find $\sum_{k=1}^{n}k^2$.

Section 1.4

1. (c) There exists $\varepsilon > 0$ such that for each $\delta > 0$ there is $x \in D$ such that $0 < |x - a| < \delta$ and $|f(x) - A| \geq \varepsilon$.

2. We prove the contrapositive, that is, we prove that if q is not multiple of 3, then q^2 is not a multiple of 3. Therefore, if $q = 3n+1$, then $q^2 = (3n+1)^2 = 9n^2 + 6n + 1 = 3\left(3n^2 + 2n\right) + 1$, thus not a multiple of 3. If $q = 3n+2$, then $q^2 = (3n+2)^2 = 9n^2 + 12n + 4 = 3\left(3n^2 + 4n + 1\right) + 1$, thus again not a multiple of 3.

3. Suppose p: the sum is even, and q: two positive integers have the same parity, are two statements. We need to prove $p \Leftrightarrow q$.

($\Rightarrow$) To prove $p \Rightarrow q$, we prove $\sim q \Rightarrow \sim p$. Thus, assume that positive integers t and s are not of the same parity. Thus, one of them, say t, is odd and the other, that is s, is even. Therefore, there exist nonnegative integers m and n such that $t = 2m+1$ and $s = 2n$. Then, $t+s = 2(m+n)+1$ is odd. Hence, $\sim p$ holds, which proves that $p \Rightarrow q$.

($\Leftarrow$) Suppose two positive integers t and s have the same parity. Then t and s are both even or both odd. If t and s are both even, there exist positive integers m and n such that $t = 2m$ and $s = 2n$. Then, $t+s = 2(m+n)$ is even. If t and s are both odd, there exist nonnegative integers m and n such that $t = 2m+1$ and $s = 2n+1$. Again, $t+s = 2(m+n+1)$ is even. So, in both cases the sum of t and s is even.

4. (a) Suppose $\sqrt{3} = \dfrac{p}{q}$, where p and q are integers with no common factors. Then, $3 = \dfrac{p^2}{q^2}$. Hence, $p^2 = 3q^2$. So, p^2 is a multiple of 3. By Exercise 2, p is a multiple of 3. Thus, there exists an integer k, such that $p = 3k$. This gives $p^2 = 9k^2$. Now, since $p^2 = 3q^2$, combining we have that $9k^2 = 3q^2$, which reduces to $q^2 = 3k^2$. Therefore, q^2 is a multiple of 3 implying that q is a multiple of 3. But this contradicts the assumption that p and q have no common factors.

(b) Suppose $\sqrt{6} = \dfrac{p}{q}$, where p and q are integers with no common factors. Then, $6 = \dfrac{p^2}{q^2}$. Hence, $p^2 = 6q^2$. So, p^2 is a multiple of 2. Therefore, p is a multiple of 2. Continue the proof similarly to the above.

(c) First we prove that if p is odd then p^3 is odd. To this end, we write $p = 2r+1$, for some integer r. Then, $p^3 = (2r+1)^3 = 8r^3 + 12r^2 + 6r + 1 = 2\left(4r^3 + 6r^2 + 3r\right) + 1$, which is odd. Next, suppose $\sqrt[3]{2} = \dfrac{p}{q}$, where $(p,q) = 1$. Then, $p^3 = 2q^3$. So, p^3 is even, which implies p is even, by the contrapositive of the first statement we proved. Thus, there exists an integer k such that $p = 2k$. This gives $p^3 = 8k^3$. Now, since $p^3 = 2q^3$, combining we have that $2q^3 = 8k^3$, which implies that q^3 is even, and thus, q even. Contradiction since $(p,q) = 1$. Hence, $\sqrt[3]{2}$ is irrational.

(d) Suppose $\sqrt{2} + \sqrt{3} = r$, r rational. Squaring and simplifying we get $2 + 2\sqrt{6} + 3 = r^2$. Therefore, $\sqrt{6} = \dfrac{r^2 - 5}{2}$, that is, $\sqrt{6}$ is rational. This is a contradiction to part (b).

7. Assume Theorem 1.3.2 holds and let S be some nonempty subset of N. We wish to prove that S has a least element, that is, we wish to show that there exists $s \in S$ such that $s \leq x$ for all $x \in S$. We prove this by assuming to the contrary that S has no least element. Note that $1 \notin S$ since otherwise it would be the least element of S. Now, define the set T by $T = \{n \in N \mid n < x \text{ for all } x \in S\}$, and let $P(n)$ be the statement "$n \in T$." Observe that $P(1)$ is true since $1 < x$ for all $x \in S$. Next, suppose $P(k)$ is true for some $k \in N$, that is, $k \in T$. Thus, $k < x$ for all $x \in S$. We will prove that $P(k+1)$ is true by assuming to the contrary that $k+1 \notin T$. Therefore, there exists $t_0 \in S$ such that $t_0 \leq k+1$. Since S has no least element, there exists $t_1 \in S$ such that $t_1 < t_0$. Since t_0 and t_1 are integers, this gives $t_1 < k+1$ and $t_1 \leq k$, which contradicts the fact that $k < x$ for all $x \in S$. Hence, $k+1 \in T$ and so $P(k+1)$ is true. Thus, by the mathematical induction, $P(n)$ is true for all $n \in N$, and so, $T = N$. Now, since we assumed that $S \neq \phi$, there exists $s_0 \in S$. But, since s_0 is a positive integer, $s_0 \in T$, which implies $s_0 < s_0$. This contradiction proves that S must have a least element.

8. Suppose $S = \{n \in N \mid P(n) \text{ is false}\}$. We will assume $S \neq \phi$ in order to get a contradiction. Thus, if $S \neq \phi$, by the well-ordering principle, S has a least element, say, t. By hypothesis (a), $P(n)$ is true for every integral power of 2. Therefore, there exists some $s \in N$ such that $2^s > t$. Let $d = 2^s - t$. Now, if $P(t+1)$ is true, by hypothesis (b), we would have $P(t)$ true, which is not the case. But, if $P(t+1)$ is false, then $P(t+2)$ is false, as well as $P(t+4), \ldots, P(t+d)$. However, $t + d = 2^s$, meaning that $P\left(2^s\right)$ is false, a contradiction. Hence, $S = \phi$ and thus, $P(n)$ is true for all $n \in N$.

Section 1.5

1. Show that $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$.

5. Let $f(x) = y$. f odd implies that $f(-x) = -f(x) = -y$. Since f^{-1} exists, $f(x) = y$ implies that $f^{-1}(y) = x$. Thus, $f(-x) = -y \Rightarrow f^{-1}(-y) = -x = -f^{-1}(y)$. Therefore, f^{-1} is odd. This statement is true if $\Re$ is replaced by an interval $(-a, a)$.

6. Even functions are not one-to-one, thus not invertible. Odd functions that are one-to-one are invertible.

7. Show that $f^{-1} = f$ and that $g^{-1} \neq g$.

8. Yes, since $f^{-1} = g$.

10. Result follows from Exercises 22(c) and 22(d) of Section 1.2.

11. We only need to prove that if $f : A \to B$ is a bijection, then $f^{-1} : B \to A$ is a bijection. Why? Let $y_1, y_2 \in B$ with $f^{-1}(y_1) = f^{-1}(y_2)$. By Theorem 1.5.4, $y_1 = f\left(f^{-1}(y_1)\right) = f\left(f^{-1}(y_2)\right) = y_2$. Therefore, f^{-1} is an injection.

If $x \in A$, then $f(x) \in B$. By Theorem 1.5.4, $x = f^{-1}(f(x))$. Therefore, f^{-1} is a surjection. Hence, f^{-1} is a bijection.

12. $g \circ f$ is a bijection by Exercise 13 of Section 1.2. We will prove that if $h(x) = (g \circ f)(x)$, then $h^{-1}(x) = \left(f^{-1} \circ g^{-1}\right)(x)$. This will prove the desired expression. To this end, we will show that $\left(h \circ h^{-1}\right)(x) = x = \left(h^{-1} \circ h\right)(x)$. So we write, $\left(h \circ h^{-1}\right)(x) = h\left(h^{-1}(x)\right) = h\left(\left(f^{-1} \circ g^{-1}\right)(x)\right) =$ $h\left(f^{-1}\left(g^{-1}(x)\right)\right) = (g \circ f)\left(f^{-1}\left(g^{-1}(x)\right)\right) = g\left(f\left(f^{-1}\left(g^{-1}(x)\right)\right)\right) = g\left(g^{-1}(x)\right) = x$. Similarly, $\left(h^{-1} \circ h\right)(x) = x$.

14. **(d)** Let $z = \arcsin x$. Then, $\sin z = x$ with $z \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Also, $\sin^2 z + \cos^2 z = 1 \Rightarrow \cos z = \sqrt{1 - x^2}$. Note, $\cos z \geq 0$ since $z \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Therefore, $\sin 2(\arcsin x) = \sin 2z = 2 \sin z \cos z = 2x\sqrt{1 - x^2}$.

15. **(a)** $\arctan\left(\tan\frac{3\pi}{4}\right) = \arctan\left[\tan\left(\frac{\pi}{4} + \pi\right)\right] = \arctan\left(\tan\frac{\pi}{4}\right) = \frac{\pi}{4}$.

(b) $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Section 1.6

2. Define $f:[0,1] \to [a,b]$ by $f(x) = a + (b-a)x$.

4. To prove part (c) of Theorem 1.6.2, suppose $A \sim B$ and $B \sim C$. This means that there exist functions $f: A \to B$ and $g: B \to C$ which are bijections. To show $A \sim C$, we need to find a bijection h from A to C. Let us choose $h = g \circ f$. Then, $h: A \to C$ and by Exercise 13 in Section 1.2, h is a bijection.

5. Here is an outline of a proof by contradiction. Suppose N is finite. By induction prove that a nonempty finite subset of $\Re$ contains a maximum element (and a minimum element). Now prove by contradiction that N is unbounded.

6. Suppose A is countable and $B \subseteq A$. We will prove that B is countable. If B is finite, then B is countable. Thus, we assume that B is infinite. In this case, A must be countably infinite. So, $N \sim A$. Let $f: N \to A$ be a bijection with the range $f(N) = \{x_n \mid n \in N\}$. Let n_1 be the smallest positive integer such that $x_{n_1} \in B$. Let n_2 be the smallest positive integer greater than n_1 such that $x_{n_2} \in B$. By induction we have that $B = \{x_{n_1}, x_{n_2}, \ldots\}$. Hence, $N \sim B$ which completes the proof.

8. **(a)** Define $f:(0,1) \to (-1,1)$ by $f(x) = 2x - 1$.

(b) Define $g:(-1,1) \to \Re$ by $g(x) = \tan \frac{\pi}{2} x$.

(c) Use Theorem 1.6.2, part (c).

9. Recall that Q is countable. To prove that $\Re \setminus Q$ is uncountable, assume to the contrary. Then $Q \cup (\Re \setminus Q)$ is countable by part (b) of Theorem 1.6.7. But this is a contradiction because $\Re$ is uncountable.

Section 1.7

1. To prove the uniqueness of the multiplicative identity, which exists by Axiom (A6), suppose there are two, say 1_1 and 1_2. Therefore, $a \cdot 1_1 = a \cdot 1_2 = a$ for all $a \in F$. Thus, if $a = 1_2$ we have $1_2 \cdot 1_1 = 1_2$, and if $a = 1_1$ we have $1_1 \cdot 1_2 = 1_1$. Due to Axiom (A2) we have $1_1 = 1_2$, meaning that the multiplicative identity is unique.

To prove part (b) of Theorem 1.7.2, observe that F contains an additive and multiplicative inverses of a. To prove an additive inverse of a is unique, we assume a has two additive inverses, say u and v. Then $u + a = 0$ and $v + a = 0$. Thus, $u = 0 + u = (v + a) + u = v + (a + u) = v + 0 = v$. Since $u = v$, the additive inverse must be unique.

To prove that $(-1)a = -a$, start with $1 + (-1) = 0$. Then, $[1 + (-1)]a = 0 \cdot a \Leftrightarrow a + (-1)a = 0 \Leftrightarrow a + (-1)a + (-a) = 0 + (-a) \Leftrightarrow a + (-a) + (-1)a = -a \Leftrightarrow 0 + (-1)a = -a \Leftrightarrow (-1)a = -a.$ Or, simply argue that since $a + (-1)a = 1 \cdot a + (-1) \cdot a = (1 + (-1))a = 0 \cdot a = 0$, the conclusion follows by part (a) of Theorem 1.7.2.

To prove part (e), note that since $0 = 0 \cdot a = a[b + (-b)] = ab + a(-b)$, $a(-b))$ is an additive inverse of ab. But, so is $-(ab)$. Since additive inverses are unique, we have $a(-b) = -(ab)$. Similarly we can prove the second equality.

By definition $-(-a)$ is the additive inverse of $-a$. Also, a is the additive inverse of $-a$. Due to uniqueness, $a = -(-a)$.

Suppose $ab = 0$. If $a = 0$, there is nothing to prove. If $a \neq 0$ then a has the multiplicative inverse $\frac{1}{a}$. Thus, $\frac{1}{a}(ab) = \frac{1}{a}(0) \Leftrightarrow \left[\frac{1}{a}(a)\right]b = 0 \Leftrightarrow 1 \cdot b = 0 \Leftrightarrow b = 0$.

2. **(a)** Since $0 = (-a)\cdot 0 = (-a)[b+(-b)] = (-a)b + (-a)(-b)$, $(-a)(-b)$ is the additive inverse of $(-a)b$. But, $(-a)b = -(ab)$. So, $(-a)(-b)$ and ab are additive inverses of $(-a)b$. Due to uniqueness, they are equal.

(b) $a+b = c+b \Leftrightarrow (a+b)+(-b) = c+b+(-b) \Leftrightarrow a+[b+(-b)] = c+[b+(-b)] \Leftrightarrow a+0 = c+0 \Leftrightarrow a = c.$

(c) $-(a+b) = (-1)(a+b) = (-1)a + (-1)b = -a-b.$

(d) Since $a \neq 0$, $\frac{1}{a}$ exists. Suppose that $\frac{1}{a} = 0$. Then we have $1 = a\cdot\frac{1}{a} = a\cdot 0 = 0$, which contradicts the uniqueness of the additive identity. Suppose that $\frac{1}{a} \neq 0$. Then by Axiom (A7), $\frac{1}{\frac{1}{a}}$ is a multiplicative inverse of $\frac{1}{a}$. Since a is also a multiplicative inverse of a, by uniqueness the desired equality holds.

3. **(a)** Suppose $a > 0$. Then, $a + (-a) > 0 + (-a)$, which gives $0 > -a$, and thus $-a < 0$.

(b) By Axiom (A9) we have 3 possibilities: $0 = 1$, $0 < 1$, or $0 > 1$. Due to uniqueness of the additive identity, $0 = 1$ is not a possibility. Suppose $0 > 1$. Then by part (a) we have $0 < -1$, which gives $0(-1) < (-1)(-1)$. Thus, $0 < 1$. But this is contradictory to our assumption. Hence, the only possibility is that $0 < 1$.

(c) According to Theorem 1.7.2, part (c), since $ab > 0$, neither a or b is zero. So, by Axiom (A9), either $a > 0$ or $a < 0$. Suppose $a > 0$. Then by Theorem 1.7.4, part (d), $\frac{1}{a} > 0$. Therefore, $b = 1\cdot b = \left(\frac{1}{a}\cdot a\right)b = \frac{1}{a}\cdot(ab) > 0$. Hence, $b > 0$. Next, suppose $a < 0$. Then, $\frac{1}{a} < 0$ and $b = \left(\frac{1}{a}\cdot a\right)b = \frac{1}{a}\cdot(ab)$ < 0. Hence, $b < 0$.

(e) Suppose $0 < a < b$. Then, by Theorem 1.7.4, part (d), $\frac{1}{a} > 0$ and $\frac{1}{b} > 0$. Suppose $\frac{1}{a} < \frac{1}{b}$. Then, $\frac{1}{a}(ab) < \frac{1}{b}(ab)$ which gives $b < a$, a contradiction. Hence, $\frac{1}{b} < \frac{1}{a}$.

4. Suppose $a, b, c \in F$ such that $a < b$ and $c < 0$. We prove that $ac > bc$. Since $c < 0$, then $-c > 0$. Thus, by Axiom (A12) we have $a(-c) < b(-c)$ which yields $-ac < -bc$. By part (a) of Theorem 1.7.4 we have $-(-ac) > -(-bc)$, which by part (f) of Theorem 1.7.2 yields the desired result.

To prove part (c) of Theorem 1.7.4, suppose a is a nonzero element of F. By Axiom (A9) we have $a < 0$ or $a > 0$. Suppose $a < 0$. Then, $-a > 0$ and by Axiom (A12) we get $(-a)(-a) > 0(-a) = 0$. But, $(-a)(-a) = (-1)(a)(-1)(a) = (-1)(-1)(a)(a) = a^2$. Thus, $a^2 > 0$. Next, suppose $a > 0$. Then by Axiom (A12), $(a)(a) > 0(a) = 0$. Again, $a^2 > 0$.

Suppose $a > 0$. Then, by Axiom (A9), $a \neq 0$. Therefore, by Axiom (A7), $\frac{1}{a} \neq 0$. Suppose that $\frac{1}{a} < 0$. Then, by part (b) of Theorem 1.7.4, we have $a\left(\frac{1}{a}\right) < 0\left(\frac{1}{a}\right) = 0$. But this yields $1 < 0$, a contradiction. Therefore, $\frac{1}{a} > 0$.

5. The result follows from Axiom (A12) and Exercise 3(e).

6. We need to show Axioms (A1) through (A12) are satisfied. Verification of Axioms (A1)–(A8) is left to the reader. The order relations are based on "$<$." We define $a < b$ for $a, b \in Q$ by requiring existence of $c \in Q$ so that $a + c = b$. With this in mind the reader can prove that Axioms (A9)–(A12) are also satisfied.

7. Let $T = \{-s \mid s \in S\}$. If m is a lower bound of S, then $s \geq m$ for all $s \in S$. Thus, $-s \leq -m$ for all $s \in S$. Hence, $-m$ is an upper bound of T. Similarly, if a is an upper bound of T, then $-a$ is a lower bound of S. Now, by Axiom (A13), T has a least upper bound, say k. We will prove $-k = \inf S$. To this end, observe that $-k$ is a lower bound of S. Let r be any other lower bound of S. Then $-r$ is an upper bound of T and $k \le -r$, since $k = \sup T$. Therefore, $r \le -k$ which implies that $-k$ is indeed the greatest lower bound of S.

8. Suppose M_1 and M_2 are two least upper bounds of A. Therefore, M_1 and M_2 are upper bounds of A. Now, since $M_1 = \sup A$, we have $M_1 \le M_2$. Also, since $M_2 = \sup A$, we have $M_2 \le M_1$. Hence, $M_1 = M_2$.

9. ($\Rightarrow$) Suppose $k = \sup S$ where S is a nonempty subset of $\Re$ and let $\varepsilon > 0$ be an arbitrary real number. Then, $k - \varepsilon < k$ and thus $k - \varepsilon$ is not an upper bound of S. Therefore, there exists $s \in S$ such that $k - \varepsilon < s \le k$.

($\Leftarrow$) Suppose that S is a nonempty subset of $\Re$, k is an upper bound of S, and for each $\varepsilon > 0$ there exists $s \in S$ such that $k - \varepsilon < s$. Now, suppose that $M < k$ and pick $\varepsilon = k - M$. Then, $\varepsilon > 0$ and by hypotheses there exists $s \in S$ such that $k - (k - M) < s$, that is, $M < s$. Therefore, M is not an upper bound of S. However, since M is an arbitrary number smaller than k, we must have that $k = \sup S$.

10. To prove $(a) \Rightarrow (b)$, let $z = \frac{y}{x}$. Then by part (a), there exists $n \in N$ such that $n > \frac{y}{x}$. This implies $y < nx$. To prove $(b) \Rightarrow (c)$, let $y = 1$ in part (b). Then we have $1 < nx$, which gives $\frac{1}{n} < x$. This gives part (c). To prove $(c) \Rightarrow (d)$, we assume to the contrary that N is unbounded above by some real number r, that is, $n < r$ for all $n \in N$. Then, by Exercise 3(e), $\frac{1}{r} < \frac{1}{n}$ for all $n \in N$. This is a contradiction to hypothesis in the case $x = \frac{1}{n}$. To prove $(d) \Rightarrow (e)$, consider the set $S = \{m \in N \mid x < m\}$. Since x is a fixed positive real number, by part (d) we know that $S \ne \phi$. Let n be the least element guaranteed by the well-ordering principle. Then, $n - 1 \notin S$ and hence, (e) follows. To prove $(e) \Rightarrow (a)$, we observe that the second inequality in (e) is indeed what (a) states. Hence, all statements are equivalent.

11. Suppose a is a rational number and b is an irrational number. To prove that ab is irrational, we assume to the contrary that ab is rational. But then, since a is rational so is $\frac{1}{a}$ and thus, $\frac{1}{a}(ab) = b$ is rational, due to the fact Q is a field. Contradiction to the hypothesis.

12. First, we prove that (a, b) contains a rational number where $a, b \in \Re$. To accomplish this we consider 3 cases. *Case 1.* Suppose $0 < a < b$. By Corollary 1.7.9, part (b), with $x = b - a$ and $y = 1$ we know that there exists $n^* \in N$ such that $1 < n^*(b - a)$, which gives $\frac{1}{n^*} < b - a$. Now define the set $S = \left\{ n \in N \,\middle|\, \frac{n}{n^*} > a \right\}$. By Theorem 1.6.8, $S \ne \phi$, and by the well-ordering principle, S has the least element, say, n_0. Then, $\frac{n_0}{n^*} > a$ and $\frac{n_0 - 1}{n^*} \le a$. Furthermore, $\frac{n_0}{n^*} \le \frac{1}{n^*} + a < (b - a) + a = b$. Hence, the rational number $\frac{n_0}{n^*} \in (a, b)$.

Case 2. Suppose $a \le 0 < b$. By Corollary 1.7.9, part (c), there exists $n^* \in N$ such that $\frac{1}{n^*} < b$. Thus, $\frac{1}{n^*}$ is a rational number in (a, b).

Case 3. Suppose $a < b \le 0$. Then, $0 \le -b < -a$, and by the preceding discussion, there exists a rational number $r \in (-b, -a)$. Hence, a rational number $-r$ is in (a, b).

Now we prove that (a, b) contains an irrational number. If $a < b$, there exists a nonzero rational number r in the interval $\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$. Why? Thus, $\frac{a}{\sqrt{2}} < r < \frac{b}{\sqrt{2}}$. So, $a < r\sqrt{2} < b$. But, by Exercise 11, $r\sqrt{2}$ is irrational, and so the proof is complete.

13. (c) $S = \left\{ x \in Q^+ \,\middle|\, x^2 \le 2 \right\}$. Note that $1 \in S$, so $S \ne \phi$. In order to prove that S has no least upper bound, we will show that the set T of all upper bounds, $T=\left\{x\in Q^+ \mid x^2>2\right\}$, has no least element. Note that $T\neq\phi$, because $2\in T$. In fact, if $a\in T$ and $b>a$, then $b^2>ab>a^2>2$, implying that $b\in T$. Now, suppose k is the least element of T. Then, k is a positive rational number, and $k^2>2$. Next, if $x=\frac{2k}{k^2-2}$, by the Archimedean order property, there exists $n\in N$ such that $x<n$. In addition, by Exercise 3(e), we have that $\frac{1}{n}<\frac{k^2-2}{2k}$. Therefore, $\frac{2k}{n}<k^2-2$, which gives $k^2-\frac{2k}{n}>2$, and hence, $k^2-\frac{2k}{n}+\frac{1}{n^2}>2$. This gives $\left(k-\frac{1}{n}\right)^2>2$, where, $k-\frac{1}{n}>k-\frac{k^2-2}{2k}=\frac{k^2+2}{2k}>0$. This means that a positive rational $k-\frac{1}{n}\in T$. However, $k-\frac{1}{n}<k$, and k is the least element of T. Contradiction. Hence, T has no least element, and thus, S has no least upper bound.

14. In order to show that Q is not a complete ordered field, we need to show, by a counterexample, that the completeness axiom does not hold in Q. To this end, choose a set, say, $S=\left\{x\in\Re\setminus Q \mid 0\le x\le\sqrt{2}\right\}$. S is bounded, nonempty, and has no least upper bound in the set of rational numbers. Or, use the set $S=\left\{x\in Q^+ \mid x^2\le 2\right\}$ from Exercise 13(c).

15. Let $S=\left\{x\in\Re^+ \mid x^2<2\right\}$. From the proof given for Exercise 13(c), we have S nonempty and bounded. Thus, by the completeness axiom, S has the least upper bound. Let $k=\sup S\in\Re^+$. We will prove that $k^2=2$ by showing that $k^2<2$ or $k^2>2$ are not possible. *Case 1.* If $k^2<2$, then $k\in S$. Also, since $1\in S$, then $k\ge 1$. We will show that there exists an element in S larger than k, which will contradict the fact that $k=\sup S$. To this end, observe that if $x=\frac{2k+1}{2-k^2}$, then, by the Archimedean order property, there exists $n\in N$ such that $x<n$. In addition, by Exercise 3(e), we have that $\frac{1}{n}<\frac{2-k^2}{2k+1}$. Thus, $\frac{1}{n}(2k+1)<2-k^2$, which implies that $\frac{2k}{n}+\frac{1}{n}<2-k^2$. Using Exercise 5, we have that $\frac{2k}{n}+\frac{1}{n^2}<2-k^2$, and so $k^2+\frac{2k}{n}+\frac{1}{n^2}<2$, which in turn gives $\left(k+\frac{1}{n}\right)^2<2$. Therefore, $k+\frac{1}{n}\in S$. But, $k=\sup S$. Thus, a contradiction. Therefore, $k^2<2$ is not a possibility.

Case 2. If $k^2>2$, then contradiction follows by the argument we used in the proof of Exercise 13(c). Hence, $k^2=2$.

16. If $x=\sqrt[3]{2}+\sqrt{3}$, then using similar steps to those in part (b) we get $x^6-9x^4-4x^3+27x^2+36x-23=0$. Since this polynomial has only integer coefficients, all real solutions are algebraic, this includes $x=\sqrt[3]{2}+\sqrt{3}$.

Section 1.8

1. Since $\frac{p}{q}$ is a root of $a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0=0$, we have $a_n\left(\frac{p}{q}\right)^n+a_{n-1}\left(\frac{p}{q}\right)^{n-1}+\cdots+a_1\left(\frac{p}{q}\right)+a_0=0$. Multiply by q^n and rearrange to obtain $a_np^n=-q\left(a_{n-1}p^{n-1}+\cdots+a_1pq^{n-2}+a_0q^{n-1}\right)$. Therefore, q divides a_np^n. Since $(p,q)=1$, q cannot divide p and so it cannot divide p^n. Thus, q divides

a_n. Similarly we can write $a_0q^n = -p\left(a_np^{n-1} + a_{n-1}p^{n-2}q + \cdots + a_2pq^{n-2} + a_1q^{n-1}\right)$ and verify that p must divide a_0.

2. Suppose $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$, with $a_n \neq 0$ and $c \in \Re$. Then, $\frac{f(x)}{x-c} = p(x) + \frac{r(x)}{x-c}$, where degree of the polynomial r is less than n. Therefore, $f(x) = (x-c)p(x) + r(x)$. If $f(c) = 0$, then $r(c) = 0$ and so $x - c$ divides $f(x)$. If $x - c$ divides $f(x)$, then $r(x) \equiv 0$ so clearly, $f(c) = 0$.

3. (a) Suppose $f(x) = x^4 - 4x^3 + 2x^2 + 8x - 8$. By Theorem 1.8.1, the only possible choices for a rational root of f are ± 8, ± 4, ± 2, and ± 1. Since $f(2) = 0$, by Exercise 2 we know that $x - 2$ divides $f(x)$. Indeed, long or synthetic division gives $f(x) = (x-2)\left(x^3 - 2x^2 - 2x + 4\right)$. Next, we factor $g(x) \equiv x^3 - 2x^2 - 2x + 4$. All of the choices for a rational root are ± 4, ± 2, and ± 1. Since $g(2) = 0$, $x - 2$ divides $g(x)$. Indeed, $g(x) = (x-2)\left(x^2 - 2\right)$. Hence, $f(x) = (x-2)^2\left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right)$. Therefore, $f(x) = 0$ if $x = 2, 2, \sqrt{2}$, and $-\sqrt{2}$. The value $x = 2$ is a "double" root.

(b) Suppose $f(x) = 9x^3 - 30x^2 + 28x - 8$. By Theorem 1.8.1, the only possibilities for a rational root of f are $\frac{p}{q}$ where p is a divisor of -8 and q is a divisor of 9. This gives $\pm\frac{8}{9}, \pm\frac{8}{3}, \pm 8, \pm\frac{4}{9}, \pm\frac{4}{3}, \pm 4, \pm\frac{2}{9}, \pm\frac{2}{3}, \pm 2, \pm\frac{1}{9}, \pm\frac{1}{3}$, and ± 1. Since $f(2) = 0$, $x - 2$ divides $f(x)$. Thus, $f(x) = (x-2)\left(9x^2 - 12x + 4\right)$, which gives $f(x) = (x-2)(3x-2)^2$. Hence, $f(x) = 0$ if $x = 2$ or $\frac{2}{3}$.

4. (a) $x^4 + 4 = x^4 + 4x^2 + 4 - 4x^2 = \left(x^2 + 2\right)^2 - 4x^2 = \left(x^2 + 2 - 2x\right)\left(x^2 + 2 + 2x\right)$

(b) Since $p(x) = 4x^2 - 2x - 1$ does not readily factor, we use the quadratic formula to solve $p(x) = 0$. Since the solution is $x = \frac{1}{4}\left(1 \pm \sqrt{5}\right)$, $p(x)$ can be factored as $p(x) = 4\left[x - \frac{1}{4}\left(1+\sqrt{5}\right)\right]\left[x - \frac{1}{4}\left(1-\sqrt{5}\right)\right] = \left[2x - \frac{1}{2}\left(1+\sqrt{5}\right)\right]\left[2x - \frac{1}{2}\left(1-\sqrt{5}\right)\right]$.

(c) Use the rational root theorem.

(e) $x^7 + 1 = (x+1)\left(x^6 - x^5 + x^4 - x^3 + x^2 - x + 1\right)$

(f) $x^5 - 1 = (x-1)\left(x^4 + x^3 + x^2 + x + 1\right)$

5. (a) Since $x^2 - x \geq 6$ can be written as $(x-3)(x+2) \geq 0$, we want all the values of x which will make the terms $x - 3$ and $x + 2$ of the same sign. Thus, we solve $x - 3 \geq 0$ and $x + 2 \geq 0$ to get $x \geq 3$. Also, $x - 3 \leq 0$ and $x + 2 \leq 0$ give $x \leq -2$. The union gives the set of all the requested values.

(b) Factor the given expression to get $(x+1)(x-2)(x+3) < 0$. Now consider cases.

(d) Preferred procedure is to subtract 3 from both sides and consider cases.

6. (a) Let $x = \frac{\sqrt{2}}{\sqrt[3]{3}}$ be a rational number. Then, $\sqrt[3]{3}x = \sqrt{2}$. Next, take 6th power to get, $9x^6 = 8$. Therefore, $x = \frac{\sqrt{2}}{\sqrt[3]{3}}$ satisfies the equation $9x^6 - 8 = 0$. Thus, by Theorem 1.8.1, $\sqrt{2}$ must divide -8 and $\sqrt[3]{3}$ must divide 9. Since this is not the case, x is not rational.

(b) If $x=\sqrt[3]{2}-\sqrt{3}$, then $\sqrt[3]{2}=x+\sqrt{3}$. Cubing gives, $2=x^3+3\sqrt{3}x^2+9x+3\sqrt{3}$. So, $2-9x-x^3=3\sqrt{3}\left(x^2+1\right)$. Squaring gives, $x^6-9x^4-4x^3+27x^2-36x-23=0$. If $x=\sqrt[3]{2}-\sqrt{3}$ is rational, then it must divide -23. Since this is not the case, $\sqrt[3]{2}-\sqrt{3}$ is irrational.

7. Suppose $x=\dfrac{p}{q}$, where $p,\ q\in Z$, with $q\neq 0$. Then, x satisfies $qx-p=0$, and so it is algebraic. Converse is false. Note that $\sqrt{2}$ is not rational but it satisfies $x^2-2=0$, so it is algebraic.

8. Consider equation $x^n-a=0$. Then, $x=\sqrt[n]{a}$ is a solution of this equation. If it is rational, then by the rational root theorem, it divides a. Hence, it is an integer, say, k. Now, if $\sqrt[n]{a}$ is an integer k, then $a=k^n$.

9. (b) Consider $\sqrt{2}^{\sqrt{2}}$. *Case 1.* Suppose $\sqrt{2}^{\sqrt{2}}$ is rational. Then, choose $\alpha=\sqrt{2}$ and $\beta=\sqrt{2}$. *Case 2.* Suppose $\sqrt{2}^{\sqrt{2}}$ is irrational. Then, choose $\alpha=\sqrt{2}^{\sqrt{2}}$ and $\beta=\sqrt{2}$, because then $\alpha^{\beta}=\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}=\left(\sqrt{2}\right)^2=2$, rational.

11. (a) Only if $f(x)\equiv 0$.

(b) No. It may seem that $f(x)\equiv\sin\dfrac{1}{x}$ is oscillatory since it has infinitely many roots. But, they are not bounded.

(c) $\sin x$ and $\dfrac{\sin x}{x}$ are two examples of oscillatory functions.

12. Proof of part (b). Suppose $a,b>0$ and let the statements be (i) $a<b$, (ii) $a^2<b^2$, and (iii) $\sqrt{a}<\sqrt{b}$. We will prove (i) $\Leftrightarrow$ (ii) and (i) $\Leftrightarrow$ (iii). So, suppose $a<b$. Then, $b+a>0$ and $b-a>0$. Thus, by Axiom (A12), $(b+a)(b-a)>0$. Therefore, $b^2-a^2>0$, which implies that (ii) holds. Now suppose $a^2<b^2$. Thus, we have $b^2-a^2=(b+a)(b-a)>0$. Since $b+a>0$, by Exercise 3(c) of Section 1.7 we have $b-a>0$. Hence, (i) holds. The fact that (i) $\Leftrightarrow$ (iii) follows from the fact that (i) $\Leftrightarrow$ (ii) if we replace a by $\sqrt{a}$ and b by $\sqrt{b}$.

Proof of part (c). By Theorem 1.7.4, part (c), we have that $\left(\sqrt{a}-\sqrt{b}\right)^2\geq 0$. Note that $a,b\geq 0$. Therefore, $a-2\sqrt{ab}+b\geq 0$. Thus, the result follows.

Proof of part (d). Since $a,b\geq 0$ we have $ab\geq 0$, $a+b\geq 0$, and $a^2+b^2\leq a^2+2ab+b^2=(a+b)^2$. Now apply Theorem 1.8.4, part (b).

13. Suppose $a>b$ and pick a particular ε, say $\varepsilon=\dfrac{a-b}{2}$. Since $a-\varepsilon<b$, then $a-\dfrac{a-b}{2}<b$, which is equivalent to $a<b$. Hence, a contradiction.

14. (b) Suppose $a,b\in\mathfrak{R}$ such that $|a|>b\geq 0$. ($\Rightarrow$) If $a>0$, then $a=|a|>b$. If $a<0$, then, $-a=|a|>b$. So, $a<-b$. Now, what if $b<0$? ($\Leftarrow$) If $a\geq 0$, then $|a|=a>b$. If $a<0$, then $|a|=-a$. Since $a<-b$, we have $|a|=-a>b$.

(c) If $a\geq 0$, then $|a|=a=\sqrt{a^2}$. If $a<0$, then $|a|=-a=\sqrt{(-a)^2}=\sqrt{a^2}$.

(d) $|ab|=\sqrt{(ab)^2}=\sqrt{a^2b^2}=\sqrt{a^2}\sqrt{b^2}=|a||b|$.

(e) First we prove that $\left|\dfrac{1}{b}\right|=\dfrac{1}{|b|}$ for $b\neq 0$. If $b>0$, then, by Theorem 1.7.4, part (d), $\dfrac{1}{b}>0$. Thus,

$\left|\frac{1}{b}\right| = \frac{1}{b} = \frac{1}{|b|}$. If $b < 0$, then $\frac{1}{b} < 0$ and thus, $\left|\frac{1}{b}\right| = -\left(\frac{1}{b}\right) = \frac{1}{-b} = \frac{1}{|b|}$. Now we multiply both sides of $\left|\frac{1}{b}\right| = \frac{1}{|b|}$ by $|a|$ and apply Theorem 1.8.5, part (d), to get the desired result.

15. **(a)** Square both sides (see Theorem 1.8.4, part (b)).

(b) Square both sides.

(c) Use Exercise 14(b).

(d) Square both sides or use Theorem 1.8.5, part (b).

(f) Do not divide by 0.

(g) Solve the same way as (h).

16. $0 \le \sqrt{a^2+b^2} \le a+b \Leftrightarrow a^2+b^2 \le (a+b)^2 \Leftrightarrow \left(\sqrt{a^2}\right)^2 + \left(\sqrt{b^2}\right)^2 \le \left[\sqrt{(a+b)^2}\right]^2 \Leftrightarrow |a|^2 + |b|^2 \le |a+b|^2$.

17. **(a)** By the triangle inequality we have $|a| = |(a-b)+b| \le |a-b| + |b|$. Thus, $|a| - |b| \le |a-b|$. Or, simply, replace a by $a-b$ in the triangle inequality.

(b) Since $|b| = |(b-a)+a| \le |b-a| + |a| = |a-b| + |a|$, we have $-|a-b| \le |a| - |b|$. Therefore, using part (a), we have $-|a-b| \le |a| - |b| \le |a-b|$, which by Theorem 1.8.5, part (b), gives the desired result.

18. **(a)** By the triangle inequality for any $x, y \in \Re$, we have $|x+y| \le |x| + |y|$. If $x = a - c$ and $y = c - b$, this is equivalent to $|(a-c)+(c-b)| \le |a-c| + |c-b|$, which is what we wished to prove.

(b) Note that $c - b > 0$, $b - a > 0$, and $c - a > 0$. Thus, $|a-b| + |b-c| = |b-a| + |c-b| = (b-a) + (c-b) = c - a = |a-c|$

19. Since $|f(x)| \le M$ for all $x \in [a,b]$, by Theorem 1.8.5, part (b), we have $-M \le f(x) \le M$. Therefore, if x_1 and x_2 are any elements in $[a,b]$, then $-M \le f(x_1) \le M$ and $-M \le f(x_2) \le M$, which is equivalent to $-M \le -f(x_2) \le M$. Combining these gives the result.

21. **(a)** Since $\sum_{k=1}^{n}(\alpha a_k + \beta b_k)^2 \ge 0$ for any real values α and β, we choose $\alpha = \sum_{k=1}^{n}(b_k)^2$ and $\beta = -\sum_{k=1}^{n} a_k b_k$.

But, $\sum_{k=1}^{n}(\alpha a_k + \beta b_k)^2 = \alpha^2 \sum_{k=1}^{n}(a_k)^2 + 2\alpha\beta \sum_{k=1}^{n} a_k b_k + \beta^2 \sum_{k=1}^{n}(b_k)^2 \ge 0$. Thus,

$\left[\sum_{k=1}^{n}(b_k)^2\right]^2 \sum_{k=1}^{n}(a_k)^2 + 2\left[\sum_{k=1}^{n}(b_k)^2\right]\left(-\sum_{k=1}^{n} a_k b_k\right)\sum_{k=1}^{n} a_k b_k + \left(-\sum_{k=1}^{n} a_k b_k\right)^2 \sum_{k=1}^{n}(b_k)^2 \ge 0$. Thus,

$\left[\sum_{k=1}^{n}(b_k)^2\right]\left\{\left[\sum_{k=1}^{n}(b_k)^2\right]\left[\sum_{k=1}^{n}(a_k)^2\right] - 2\left[\sum_{k=1}^{n} a_k b_k\right]^2 + \left[\sum_{k=1}^{n} a_k b_k\right]^2\right\} \ge 0$. Since $\sum_{k=1}^{n}(b_k)^2 \ge 0$ we have

$\left[\sum_{k=1}^{n}(b_k)^2\right]\left[\sum_{k=1}^{n}(a_k)^2\right] - \left(\sum_{k=1}^{n} a_k b_k\right)^2 \ge 0$. Therefore, $\left(\sum_{k=1}^{n} a_k b_k\right)^2 \le \left[\sum_{k=1}^{n}(b_k)^2\right]\left[\sum_{k=1}^{n}(a_k)^2\right]$. Other methods of proof include induction or considering the discriminant of a polynomial $P(x) = \sum_{k=1}^{n}(a_k x - b_k)^2$.

(b) Since, using part (a), $\sum_{k=1}^{n}(a_k + b_k)^2 = \sum_{k=1}^{n}\left[(a_k)^2 + 2a_k b_k + (b_k)^2\right] = \sum_{k=1}^{n}(a_k)^2 + 2\sum_{k=1}^{n} a_k b_k + \sum_{k=1}^{n}(b_k)^2 \le$

$$\sum_{k=1}^{n}(a_k)^2 + 2\left[\sum_{k=1}^{n}(a_k)^2\right]^{\frac{1}{2}}\left[\sum_{k=1}^{n}(b_k)^2\right]^{\frac{1}{2}} + \sum_{k=1}^{n}(b_k)^2 = \left\{\left[\sum_{k=1}^{n}(a_k)^2\right]^{\frac{1}{2}} + \left[\sum_{k=1}^{n}(b_k)^2\right]^{\frac{1}{2}}\right\}^2$$, the result follows.

22. **(e)** $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

23. Rationalize the numerator in Exercise 22(e).

24. Write $f(x)$ as, $f(x) = a\left(x^2 + \dfrac{b}{a}x\right) + c = a\left(x^2 + \dfrac{b}{a}x + \dfrac{b^2}{4a^2}\right) + c - a\left(\dfrac{b^2}{4a^2}\right) = a\left(x + \dfrac{b}{2a}\right)^2 - \dfrac{b^2 - 4ac}{4a}$.

25. *Case 1.* Suppose $x < -\dfrac{b}{2a}$. The distance of x to $-\dfrac{b}{2a}$ is $-\dfrac{b}{2a} - x$. Therefore, we need to show that $f(x) = f\left(x + 2\left(-\dfrac{b}{2a} - x\right)\right)$, that is, $f(x) = f\left(-x - \dfrac{b}{2a}\right)$. But, $f\left(-x - \dfrac{b}{2a}\right) = \ldots = f(x)$.

Suppose $x > -\dfrac{b}{2a}$. The distance of x to $-\dfrac{b}{2a}$ is $x - \left(-\dfrac{b}{2a}\right)$. Therefore, we need to show that $f(x) = f\left(x - 2\left[x - \left(-\dfrac{b}{2a}\right)\right]\right)$, that is, $f(x) = f\left(-x - \dfrac{b}{a}\right)$. This holds by Case 1.

27. Complete the square to get two straight lines.

28. $a = -b$ also works.

Section 1.9

1. T	**10.** F	**19.** T	**28.** T	**37.** T	**46.** T	**55.** T
2. T	**11.** T	**20.** T	**29.** T	**38.** T	**47.** T	**56.** T
3. F	**12.** F	**21.** T	**30.** F	**39.** T	**48.** F	**57.** T
4. F	**13.** T	**22.** T	**31.** F	**40.** F	**49.** T	**58.** T
5. T	**14.** T	**23.** T	**32.** T	**41.** F	**50.** T	**59.** F
6. F	**15.** T	**24.** T	**33.** T	**42.** T	**51.** T	**60.** F
7. F	**16.** F	**25.** F	**34.** T	**43.** T	**52.** T	**61.** F
8. F	**17.** T	**26.** T	**35.** T	**44.** T	**53.** T	
9. T	**18.** T	**27.** T	**36.** T	**45.** T	**54.** T	