Chapter 2: Exponents

Introductory Problem

Products of the same base number can be represented with exponent notation as follows:

10	$= 10^1$
$10 * 10$	$= 10^2$
$10 * 10 * 10$	$= 10^3$
$10 * 10 * 10 * 10$	$= 10^4$

2	$= 2^1$
$2 * 2$	$= 2^2$
$2 * 2 * 2$	$= 2^3$
$2 * 2 * 2 * 2$	$= 2^4$

1. What exponent (?) is needed to represent $10^2 * 10^3 = 10^?$

   Answer: $(10 * 10) * (10 * 10 * 10) = 10^5$

2. What exponent (?) is needed to represent $10^4 / 10^2 = 10^?$

   Answer: $(10 * 10 * \cancel{10} * \cancel{10}) / (\cancel{10} * \cancel{10}) = 10^2$

3. What about $2^2 * 2 = 2^?$

   Answer: $(2 * 2) * (2) = 2^3$

4. What about $2^5 / 2^2 = 2^?$

   Answer: $(2 * 2 * 2 * \cancel{2} * \cancel{2}) / (\cancel{2} * \cancel{2}) = 2^3$

5. Can you discover a rule for multiplying common base numbers that have exponents?

   Answer: When multiplying common base numbers with exponents, the result will be the same base raised to the **sum** of the exponents.

   Test the rule using the base number 3.

   Answer: $3^2 * 3^2 = 3^{3+2} = 3^5$

6. Can you discover a rule for dividing common base numbers that have exponents?

   Answer: When dividing common base numbers with exponents, the result will be the same base raised to the **difference** of the exponents.

   Test the rule using the base number 3.

   $3^3 / 3^2 = 3^{3-2} = 3^1$

Practice Problems

Section 2-2

Simplify the following expressions by factoring as much as possible. Remove like pairs of factors from inside the radical. Where possible, eliminate the radical altogether. When it is not possible to eliminate the radical, leave any remaining factors inside one or more individual radicals.

2-2.1	$\sqrt{(3*3)}$			$= 3$
2-2.2	$\sqrt{(2*3*2)}$			$= 2\sqrt{3}$
2-2.3	$\sqrt{(3*3*3)}$			$= 3\sqrt{3}$
2-2.4	$\sqrt{(3*3*3*2*2)}$	$= 3 * 2\sqrt{3}$		$= 6\sqrt{3}$
2-2.5	$\sqrt{(2*2*2*2*3*3*5)}$	$= 2 * 2 * 3\sqrt{5}$		$= 12\sqrt{5}$
2-2.6	$\sqrt{(4^2)}$	$= \sqrt{(4 * 4)}$		$= 4$
2-2.7	$\sqrt{(2^2 * 3^2)}$	$= \sqrt{(2 * 2 * 3 * 3)}$	$= 2 * 3$	$= 6$
2-2.8	$\sqrt{(2^2 * 3)}$	$= \sqrt{(2 * 2 * 3)}$		$= 2\sqrt{3}$
2-2.9	$\sqrt{(a^2)}$	$= \sqrt{(a * a)}$		$= a$
2-2.10	$\sqrt{(a^2 * a)}$	$= \sqrt{(a * a * a)}$		$= a\sqrt{a}$
2-2.11	$\sqrt{(12)}$	$= \sqrt{(3 * 2 * 2)}$		$= 2\sqrt{3}$
2-2.12	$\sqrt{(24)}$	$= \sqrt{(3 * 2 * 2 * 2)}$		$= 2\sqrt{2} * \sqrt{3}$
2-2.13	$\sqrt{(60)}$	$= \sqrt{(5 * 3 * 2 * 2)}$		$= 2\sqrt{3} * \sqrt{5}$
2-2.14	$9 / \sqrt{27}$	$= 9 / \sqrt{(3 * 3 * 3)}$	$= 9 / 3\sqrt{3}$	$= 3 / \sqrt{3}$
		(Remove radical from denominator)		
		$= (3 / \sqrt{3}) * (\sqrt{3} / \sqrt{3})$	$= (3\sqrt{3}) / 3$	$= \sqrt{3}$
2-2.15	$8 / \sqrt{128}$	$= 8 / \sqrt{(2 * 2 * 2 * 2 * 2 * 2 * 2)}$		
		$= 8 / (8\sqrt{2})$		$= 1 / \sqrt{2}$
		(Remove radical from denominator)		
		$= (1 / \sqrt{2}) * (\sqrt{2} / \sqrt{2})$		$= \sqrt{2} / 2$

Section 2-3

Where possible, apply exponent rule 1 to simplify the following expressions:

2-3.1	$2^2 * 2$	$= 2^{(2+1)}$	$= 2^3$
2-3.2	$3^2 * 3^3$	$= 3^{(2+3)}$	$= 3^5$
2-3.3	$4^3 * 4^6$	$= 4^{(3+6)}$	$= 4^9$
2-3.4	$y^7 * y^8$	$= y^{(7+8)}$	$= y^{15}$
2-3.5	$2^4 * 2^8 * 3^5$	$= 2^{(4+8)} * 3^5$	$= 2^{12} * 3^5$
2-3.6	$5^4 * 5^7$	$= 5^{(4+7)}$	$= 5^{11}$
2-3.7	$3x^5 * (4y^3)$	Can not simplify because there is no common base.	
2-3.8	$5y^2 * y(-2y^4)$	$= (5)(-2)\, y^{(2+1+4)}$	$= -10\, y^7$
2-3.9	$a^3 * b^2$	Can not simplify because there is no common base.	
2-3.10	$(2^3) * (2^3)$	$= 2^{(3+3)}$	$= 2^6$

Where possible, apply exponent rule 2 to simplify the following expressions:

2-3.11	$2^2 / 2$	$= 2^{(2-1)}$	$= 2^1$	$= 2$	
2-3.12	$3^3 / 3^2$	$= 3^{(3-2)}$	$= 3^1$	$= 3$	
2-3.13	$4^6 / 4^5$	$= 4^{(6-5)}$	$= 4^1$	$= 4$	
2-3.14	y^8 / y^6	$= y^{(8-6)}$		$= y^2$	
2-3.15	$2^8 / 2^4$	$= 2^{(8-4)}$		$= 2^4$	also = 16
2-3.16	$5^7 / 5^3$	$= 5^{(7-3)}$		$= 5^4$	also = 625
2-3.17	$3x^5 / (4y^3)$	Can not be simplified because there is no common base.			
2-3.18	$5y^6 / (y * y^4)$	$= 5 * y^6 / y^{(1+4)}$	$= 5 * y^6 / y^5$		(Apply rule 1first)
			$= 5y^{(6-5)}$	$= 5y$	
2-3.19	a^3 / b^2	Can not be simplified because there is no common base.			
2-3.20	$(2^3) / (2^2 / 2)$	$= 2^3 / 2^{(2-1)}$	$= 2^3 / 2^1$		(Apply rule twice)
			$= 2^{(3-1)}$	$= 2^2$	also = 4

Apply exponent rule 3 to simplify the following expressions:

2-3.21	$(3^2)^3$	$= 3^{(2*3)}$	$= 3^6$	also = 729
2-3.22	$(y^3)^5$	$= y^{(3*5)}$	$= y^{15}$	
2-3.23	$(x^2)^5$	$= x^{(2*5)}$	$= x^{10}$	
2-3.24	$(a^5)^6$	$= a^{(5*6)}$	$= a^{30}$	
2-3.25	$(2^2)^3$	$= 2^{(2*3)}$	$= 2^6$	also = 64
2-3.26	$(10^3)^2$	$= 10^{(3*2)}$	$= 10^6$	
2-3.27	$(2^3)^4$	$= 2^{(3*4)}$	$= 2^{12}$	also = 4096
2-3.28	$(x^b)^4$	$= x^{(b*4)}$	$= x^{4b}$	
2-3.29	$[(x^3)^3]^4$	$= x^{(3*3*4)}$	$= x^{36}$	(Apply rule 3 twice)
2-3.30	$[(x^2)^4]^2$	$= x^{(2*4*2)}$	$= x^{16}$	(Apply rule 3 twice)

Use exponent rule 4 to give an expression with a single exponent:

2-3.31	$2^3 * 4^3$	$= (2 * 4)^3$	$= 8^3$	
2-3.32	$4^2 * 3^2$	$= (4 * 3)^2$	$= 12^2$	
2-3.33	$(3x)^3 * 2^3$	$= (3x * 2)^3$	$= (6x)^3$	
2-3.34	$a^2 * c^2$		$= (ac)^2$	
2-3.35	$(2x)^3 * (3y)^3$	$= (2x * 3y)^3$	$= (6xy)^3$	also $= 216\ (xy)^3$

Use the reverse of exponent rule 4 to expand the expressions below:

2-3.36	$(3 * 5)^4$		$= 3^4 * 5^4$	
2-3.37	$(a * b)^n$		$= a^n * b^n$	
2-3.38	$(2 * 2 * 2)^2$	(Apply rule 4 twice)	$= 2^2 * 2^2 * 2^2$	
2-3.39	$(x * y * z)^2$	(Apply rule 4 twice)	$= x^2 * y^2 * z^2$	
2-3.40	$(x^2 * y^3)^2$		$= (x^2)^2 * (y^3)^2$	also $= x^4 * y^6$

Use exponent rule 5 to give an expression with a single exponent:

2-3.41	$2^2 / 3^2$		$= (2 / 3)^2$	
2-3.42	$3^2 / 4^2$		$= (3 / 4)^2$	
2-3.43	a^3 / b^3		$= (a / b)^3$	
2-3.44	x^2 / y^2		$= (x / y)^2$	
2-3.45	$(x^2 / y^2)^2$	$= [(x / y)^2]^2$	$= (x / y)^4$	(Apply rule 4 first)

Use the reverse of exponent rule 5 to expand the following expressions:

2-3.46 $(2 / 3)^3$ $= 2^3 / 3^3$
2-3.47 $(5/ 7)^2$ $= 5^2 / 7^2$
2-3.48 $(3 / 4)^n$ $= 3^n / 4^n$
2-3.49 $(2x / 5)^2$ $= (2x)^2 / 5^2$ $= 4x^2 / 5^2$
2-3.50 $(3 / 2y)^3$ $= 3^3 / (2y)^3$ $= 3^3 / 8y^3$

Apply the rules for reciprocals to the following problems:

2-3.51 Take the reciprocal of 5. = 1/5
2-3.52 Take the reciprocal of $3x$. $= 1 / (3x)$
2-3.53 Take the reciprocal of x / y. $= y / x$
2-3.54 Take the reciprocal of 1/5. = 5
2-3.55 Take the reciprocal of 1/6. = 6
2-3.56 Multiply 4 by the reciprocal of 2 = 4 * (1/2) = 2
2-3.57 Take the reciprocal of the reciprocal of 3. = 1 / (1/3) = 3
2-3.58 Express the reciprocal of 8/7 as a fraction. = 7/8
2-3.59 Express the reciprocal of 5/3 as a fraction. = 3/5
2-3.60 Multiply the number 12,345 by its reciprocal. = 12345 * (1 / 12345) = 1

Determine the reciprocal of the following expressions by flipping the sign of the exponent. Remember that expressions without an exponent have an implied exponent of 1.

2-3.61 5 $= 5^{-1}$
2-3.62 3^2 $= 3^{-2}$
2-3.63 $(x + y)^2$ $= (x + y)^{-2}$
2-3.64 5^{-1} $= 5$
2-3.65 $(x + y)^{-2}$ $= (x + y)^2$

Use the double flip method to write equivalent expressions having only positive exponents.

2-3.66 3^{-4} $= 1 / 3^4$
2-3.67 $1 / 3^{-2}$ $= 3^2$
2-3.68 $3y / x^{-2}$ $= 3yx^2$
2-3.69 $2a^2b^{-1}$ $= 2a^2 / b$
2-3.70 $(2x)^{-2}$ $= 1 / (2x)^2$

Remove any negative signs from inside parentheses according to the preceding rule:

2-3.71 $(-5)^1$ $= -5$
2-3.72 $(-2x)^3$ $= -(2x)^3$
2-3.73 $(-100)^2$ $= 100^2$
2-3.74 $1 / (-3)^2$ $= 1 / 3^2$
2-3.75 $(-x)^{-3} * (-y)^{-3}$ $= x^{-3}y^{-3}$
2-3.76 $(x)^3 * (-y)^3$ $= -(x^3y^3)$

Where possible, apply the rules of exponents to simplify the following expressions to a base number with a single exponent. Note: there are three parts to each problem.

2-3.77 $10^{\frac{1}{2}} * 10^{\frac{1}{2}} = 10^{(\frac{1}{2}+\frac{1}{2})} = 10^1$
$10^{\frac{1}{3}} * 10^{\frac{1}{3}} * 10^{\frac{1}{3}} = 10^{(\frac{1}{3}+\frac{1}{3}+\frac{1}{3})} = 10^1$
$10^{\frac{1}{4}} * 10^{\frac{1}{4}} * 10^{\frac{1}{4}} * 10^{\frac{1}{4}} = 10^{(\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4})} = 10^1$

2-3.78 $10^{\frac{1}{2}} / 10^{\frac{1}{2}} = 10^{(\frac{1}{2}-\frac{1}{2})} = 10^0$
$10^{\frac{1}{3}} / 10^{\frac{1}{3}} = 10^{(\frac{1}{3}-\frac{1}{3})} = 10^0$
$10^{\frac{1}{4}} / 10^{\frac{1}{4}} = 10^{(\frac{1}{4}-\frac{1}{4})} = 10^0$

2-3.79 $2^2 * 2^{\frac{1}{2}} = 2^{(2+\frac{1}{2})} = 2^{5/2}$
$2^{\frac{1}{2}} * 2^{\frac{1}{3}} = 2^{(\frac{1}{2}+\frac{1}{3})} = 2^{\frac{5}{6}}$
$2^{\frac{1}{2}} * \sqrt{2} = 2^{\frac{1}{2}} * 2^{\frac{1}{2}} = 2^{(\frac{1}{2}+\frac{1}{2})} = 2^1$

2-3.80 $10^{\frac{1}{2}} * 10^{\frac{1}{3}} = 10^{(\frac{1}{2}+\frac{1}{3})} = 10^{\frac{5}{6}}$
$10^{\frac{1}{2}} / 10^{\frac{1}{3}} = 10^{(\frac{1}{2}-\frac{1}{3})} = 10^{\frac{1}{6}}$
$10^{\frac{1}{2}} * 10 = 10^{\frac{1}{2}} * 10^1 = 10^{(\frac{1}{2}+1)} = 10^{\frac{3}{2}}$

2-3.81 $(2^{\frac{1}{2}})^2 = 2^{\frac{1}{2} * 2} = 2^1$
$(\sqrt{2})^2 = 2^{\frac{1}{2} * 2} = 2^1$
$2^{\frac{1}{2}} * 3^{\frac{1}{2}} = (2 * 3)^{\frac{1}{2}} = 6^{\frac{1}{2}}$ (By applying rule 4 in reverse)

2-3.82 2^(1/2) $= 2^{\frac{1}{2}}$
2^0.5 $= 2^{\frac{1}{2}}$
2^1.5 $= 2^{\frac{3}{2}}$

2-3.83 $(7^{\frac{1}{2}})^2 = 7^{\frac{1}{2} * 2} = 7^1$
$(6^{\frac{1}{2}})^2 = 6^{\frac{1}{2} * 2} = 6^1$
$(5^{\frac{1}{2}})^2 = 5^{\frac{1}{2} * 2} = 5^1$

2-3.84 $2^{\frac{1}{2}} + 2^{\frac{1}{2}}$

Rules of exponents do not apply to addition of like bases.
However, by chance this can be written as twice $2^{\frac{1}{2}}$.
$2(2^{\frac{1}{2}})$ can be simplified as: $= 2^1 * 2^{\frac{1}{2}} = 2^{\frac{1}{2}+1} = 2^{\frac{3}{2}}$

$2^{\frac{1}{2}} * (1/2^{\frac{1}{2}}) = 2^{\frac{1}{2}} * (2^0 / 2^{\frac{1}{2}}) = 2^{\frac{1}{2}+0-\frac{1}{2}} = 2^0$
$1 / 2^{\frac{1}{2}} = 2^0 / 2^{\frac{1}{2}} = 2^{0-\frac{1}{2}} = 2^{-\frac{1}{2}}$

Use the rules of exponents to combine like bases and simplify the following expressions:

2-3.85	$(2^3 * 5^3 * 2^{-2} * 5) / (5^6 * 5^{-2} * 2)$	
	$[(2^3 * 2^{-2}) / (2^1)] * [(5^1 * 5^3) / (5^6 * 5^{-2})]$	Grouping like bases (*Note*: $2 = 2^1$)
	$(2^{3+(-2)} / 2^1) * (5^{1+3} / 5^{6+(-2)})$	Adding exponents for multiplied like bases
	$(2^1 / 2) * (5^4 / 5^4)$	Simplifying
	$(2^{1-1}) * (5^{4-4})$	Subtracting exponents for divided like bases
	$2^0 * 5^0$	Simplifying
	$1 * 1$	Any non zero number to the zeroeth power is 1
	$\mathbf{1}$	The solution
2-3.86	$(10^3 * 3^2 * \sqrt{3}) * (10^2 / 3^{\frac{1}{2}})$	
	$(10^3 * 10^2) * (3^2 * 3^{\frac{1}{2}} / 3^{\frac{1}{2}})$	Grouping like bases (Note: $\sqrt{3} = 3^{\frac{1}{2}}$)
	$(10^{3+2}) * (3^{2+\frac{1}{2}} / 3^{\frac{1}{2}})$	Adding exponents for multiplied like bases
	$10^5 * (3^{2\frac{1}{2}} / 3^{\frac{1}{2}})$	Simplifying
	$10^5 * (3^{2\frac{1}{2}-\frac{1}{2}})$	Subtracting exponents for divided like bases
	$\mathbf{10^5 * 3^2}$	Simplifying to get the solution
2-3.87	$(10^2 * 10^4 * 10^3 * 10) / (10^2 * 10^5 * 10^4)$	
	$(10^2 * 10^4 * 10^3 * 10^1) / (10^2 * 10^5 * 10^4)$	Already grouped (Note: $10 = 10^1$)
	$(10^{2+4+3+1}) / (10^{2+5+4})$	Adding exponents for multiplied like bases
	$10^{10} / 10^{11}$	Simplifying
	10^{10-11}	Subtracting exponents for divided like bases
	$\mathbf{10^{-1}}$	Simplifying to get the solution
2-3.88	$(2^2 * 3^2 * 2^3 * 3^5 * 2^4 * 3^{-4}) / (2^5 * 3)$	
	$(2^2 * 2^3 * 2^4 / 2^5) * (3^2 * 3^5 * 3^{-4} / 3)$	Grouping like bases
	$(2^{2+3+4} / 2^5) * (3^{2+5-4} / 3^1)$	Adding exponents for multiplied like bases
	$(2^9 / 2^5) * (3^3 / 3^1)$	Simplifying
	$(2^{9-5}) * (3^{3-1})$	Subtracting exponents for divided like bases
	$\mathbf{2^4 * 3^2}$	Simplifying to get the solution
2-3.89	$(2^{\frac{1}{2}} * 2^{\frac{1}{2}} * 3^{\frac{1}{2}} * 3^2) / (3^{-\frac{1}{2}} * 2)$	
	$(2^{\frac{1}{2}} * 2^{\frac{1}{2}} / 2^1) * (3^{\frac{1}{2}} * 3^2 / 3^{-\frac{1}{2}})$	Grouping like bases (Note: $2 = 2^1$)
	$(2^{\frac{1}{2}+\frac{1}{2}} / 2^1) * (3^{2\frac{1}{2}} / 3^{-\frac{1}{2}})$	Adding exponents for multiplied like bases
	$(2^1 / 2^1) * (3^{2\frac{1}{2}} / 3^{-\frac{1}{2}})$	Simplifying
	$(2^{1-1}) * (3^{2\frac{1}{2}-(-\frac{1}{2})})$	Subtracting exponents for divided like bases
	$2^0 * 3^3$	Simplifying
	$1 * 3^3$	Since $2^0 = 1$
	$\mathbf{3^3}$	The solution

Section 2-4

Convert the following numbers from decimal notation to scientific notation. Round the result to *five significant digits*:

2-4.1	123456.7	$= 1.2346 * 10^5$
2-4.2	12345.67	$= 1.2346 * 10^4$
2-4.3	1234.567	$= 1.2346 * 10^3$

2-4.4	123.4567	$= 1.2346 * 10^2$
2-4.5	12.34567	$= 1.2346 * 10^1$
2-4.6	1.234567	$= 1.2346 * 10^0$
2-4.7	0.1234567	$= 1.2346 * 10^{-1}$
2-4.8	0.01234567	$= 1.2346 * 10^{-2}$
2-4.9	0.001234567	$= 1.2346 * 10^{-3}$
2-4.10	100000	$= 1.0000 * 10^5$
2-4.11	1000000000	$= 1.0000 * 10^9$
2-4.12	1000000001	$= 1.0000 * 10^9$
2-4.13	0.000001234567	$= 1.2346 * 10^{-6}$
2-4.14	0.00000123	$= 1.2300 * 10^{-6}$

Convert the following numbers from scientific notation to decimal notation.
Maintain the same number of significant figures.

2-4.15	$3.21 * 10^4$	= 32100
2-4.16	$3.2 * 10^{-3}$	= 0.0032
2-4.17	$3.210 * 10^{-3}$	= 0.003210
2-4.18	$1.57 * 10^5$	= 157000
2-4.19	$9.99 * 10^1$	= 99.9
2-4.20	$8.123 * 10^{-2}$	= 0.08123

Evaluate the following expressions by writing each number in scientific notation. Combine the decimal part and their exponents to express the result in scientific notation rounded to four significant digits.

2-4.21 10,002,000 * 0.000234 / 1.5678

$= [(1.0002 * 10^7) * (2.34 * 10^{-4}] / (1.5678 * 10^0)$

$= 1.492835821 * 10^{(7-4-0)}$

$= \mathbf{1.493 * 10^3}$ (rounded to four significant digits)

2-4.22 2.5678 * 0.00001987 / 234,000

$= [(2.5678 * 10^0) * (1.987 * 10^{-5})] / (2.34 * 10^5)$

$= 2.180435299 * 10^{(0-5-5)}$

$= \mathbf{2.180 * 10^{-10}}$ (rounded to four significant digits)

2-4.23 0.00024689 * 0.00000001 * 58,000,000,000

$= (2.4689 * 10^{-4}) * (1.0 * 10^{-8}) * (5.8 * 10^{10})$

$= 14.31962 * 10^{(-4-8+10)}$

$= 14.31962 * 10^{-2}$

$= 1.431962 * 10^{-1}$ (shift decimal point to make value between 1 and 10)

$= \mathbf{1.432 * 10^{-1}}$ (rounded to four significant digits)

2-4.24 365.1 * (½) * 0.00001 * 0.0000023 / 0.00000000045

$= [(3.651 * 10^2) * (5.0 * 10^{-1}) * (1.0 * 10^{-5}) * (2.3 * 10^{-6})] / (4.5 * 10^{-10})$

$= 9.330333 * 10^{[2-1-5-6-(-10)]}$

$= 9.330333 * 10^0$

$= \mathbf{9.330 * 10^0}$ (rounded to four significant digits)

2-4.25 This famous problem involves placing grains of wheat on the squares of a chessboard. A single grain of wheat is placed on the first square. Two grains are placed on the next square and four grains on the next. The board is filled with wheat, each square receiving twice that of the preceding square, until all 64 squares are filled. Of course, the number of grains of wheat become far too great to fit on a square, or on the entire board, or even the room that holds the board.

a) Express the number of grains on the first square as 2 raised to some power.

One grain belongs on the first square. The number one can be expressed as 2^0

b) Express the number of grains that belong on the final square as 2 raised to a power.

The exponent associated with each square is one less than the number of the square. **Therefore, 2^{63} grains belong on the 64th square.**
Of course they won't all fit. There isn't even that number of grains in the world, or even in the entire universe.

c) Express the total number of grains that belong on the board in terms of 2 raised to some power.

The sum of grains on the board is: $2^0 + 2^1 + 2^2 + \ldots 2^{63}$

This is such a large number it is not possible to compute it directly.
However, by computing the sum for just the first square, and then the first two squares and the first three squares, and so on, a pattern will emerge.

Sum of first square	1	$= 1$
Sum of first 2 squares	$1 + 2$	$= 3$
Sum of first three squares	$1 + 2 + 4$	$= 7$
Sum of first four squares	$1 + 2 + 4 + 8$	$= 15$

Notice that the sum of each group is one less than a power of 2 and that power is the same as the number of squares in the group:

Sum of first square	1	$= 2^1 - 1$
Sum of first 2 squares	3	$= 2^2 - 1$
Sum of first three squares	7	$= 2^3 - 1$
Sum of first four squares	15	$= 2^4 - 1$

Therefore, the sum of all 64 squares is $2^{64} - 1$.

In the next chapter, after you have learned about binary numbers, you will see that this sum can be expressed as a binary number made up of a string of 64 ones:

1111 1111 1111 1111 1111 1111 1111 1111 1111 1111 1111 1111 1111 1111 1111 1111

You will also learn that the binary number above is one less than 2^{64}, which can be written as the binary number formed by 1 followed by 64 zeros:

1 0000 0000 0000 0000 0000 0000 0000 0000 0000 0000 0000 0000 0000 0000 0000 0000

2-4.26 The National Collegiate Athletic Association (NCAA) men's Division I basketball tournament, played in the United States each March, involves the best college teams in the nation, playing an elimination-style tournament. Games are played in rounds; winning teams in each round continue, while losers are eliminated. There are the following rounds: first round, second round, regional semifinal, regional final, national semifinal, and national final. Many teams pair up to play games in the first round. Only the winners go on to play games in the second round, and only two teams survive to play in the final round. The winner of the final round becomes the national champion. From this information, determine the following:

a) How many teams begin the tournament?

There are 6 rounds with two teams playing in the final round.
The number of teams in any round is twice that in the previous round.

Therefore:	Round 6	has 2 teams
	Round 5	has 4 teams
	Round 4	has 8 teams
	Round 3	has 16 teams
	Round 2	has 32 teams
	Round 1	**has 64 teams**

Another way to look at it is there must be $2^6 = 64$ teams that start the tournament.

b) How many games are played in the tournament?

Since two teams play in each game there will be half the number of games in each round as there are teams: Therefore:

Round 6	has 1 game
Round 5	has 2 games
Round 4	has 4 games
Round 3	has 8 games
Round 2	has 16 games
Round 1	has 32 games

The total number of games can be found by adding the six terms:

$$\mathbf{1 + 2 + 4 + 8 + 16 + 32 = 63 \text{ games.}}$$

Another way to look at it is that all 64 teams that start the tournament loose exactly one game, except for the national champion. Thus, since each loss represents one game, there are $64 - 1 = 63$ games.

Note to Instructor: Please inform your students that this problem is based on the tournament in place when the first edition was written. The tournament for 2011 with an additional 8-team first round does not demonstrate exponents nearly as well.

2-4.27 The following problem is adapted from one presented as a weekly puzzler on the National Public Radio Program *Cartalk*.

Your math class is studying probability by flipping a coin and seeing how many times in a row you can call the flip correctly. Your instructor breaks the class of 32 students into teams of two persons each. One member of each team flips a coin while the other calls heads or tails. If a flip is called correctly, the caller wins; otherwise the flipper wins. You can form new teams whenever you like, and each person keeps track of his or her individual wins.

Your instructor offers a wager. Any student who can win five flips in a row will receive an A on the next quiz. But if they fail even once, they receive an F. There are no takers. So another wager is offered. An A will go to the student who can devise a method that guarantees someone in the class will win five flips in a row. Can you come up with such a method? (*Hint*: A look at problem 2-4.26 will be helpful.)

Solution: Run the contest as a tournament. Divide the class of 32 into 16 teams of two persons each. After the first flip there are 16 winners and 16 losers. The losers sit down and the winners survive to flip another time. The winners form new teams and flip again. This process is repeated until a single champion emerges who has won five times in a row.

The table below shows how it goes:

Flip	*Persons Playing*	*Winners*
1	32	16
2	16	8
3	8	4
4	4	2
5	2	1

The procedure can not predict who this person will be, but it guarantees that one person will emerge the champion.

Tip to the Instructor:

You may want to revisit this problem in Chapter 3 after you study number systems. A minor variation to the rules of flipping will permit another solution using binary numbers.

Suppose the instructor flips five times. Each student tries to call all five flips correctly. Now what method will guarantee that someone will win?

In this case you write out the 32 binary numbers from 00000 to 11111, placing each on slip of paper. Then each student draws one slip out of a hat and lets the five bits dictate their guesses. (Zero is heads and one is tails.) Because there are exactly 32 (or 2^5) possible ways to flip five times, one of the 32 students must win.