CHAPTER 1

1.1 Use calculus to solve Eq. (1.9) for the case where the initial velocity $v(0)$ is nonzero.

We will illustrate two different methods for solving this problem: (1) separation of variables, and (2) Laplace transform.

$$\frac{dv}{dt} = g - \frac{c}{m}v$$

Separation of variables: Separation of variables gives

$$\int \frac{1}{g - \frac{c}{m}v} dv = \int dt$$

The integrals can be evaluated as

$$-\frac{\ln\left(g - \frac{c}{m}v\right)}{c/m} = t + C$$

where C = a constant of integration, which can be evaluated by applying the initial condition to yield

$$C = -\frac{\ln\left(g - \frac{c}{m}v(0)\right)}{c/m}$$

which can be substituted back into the solution

$$-\frac{\ln\left(g - \frac{c}{m}v\right)}{c/m} = t - \frac{\ln\left(g - \frac{c}{m}v(0)\right)}{c/m}$$

This result can be rearranged algebraically to solve for v,

$$v = v(0)e^{-(c/m)t} + \frac{mg}{c}\left(1 - e^{-(c/m)t}\right)$$

where the first part is the general solution and the second part is the particular solution for the constant forcing function due to gravity. For the case where, $v(0) = 0$, the solution reduces to Eq. (1.10)

$$v = \frac{mg}{c}\left(1 - e^{-(c/m)t}\right)$$

Laplace transform solution: An alternative solution is provided by applying Laplace transform to the differential equation to give

$$sV(s) - v(0) = \frac{g}{s} - \frac{c}{m}V(s)$$

Solve algebraically for the transformed velocity

$$V(s) = \frac{v(0)}{s + c/m} + \frac{g}{s(s + c/m)} \qquad (1)$$

The second term on the right of the equal sign can be expanded with partial fractions

$$\frac{g}{s(s + c/m)} = \frac{A}{s} + \frac{B}{s + c/m} = \frac{A(s + c/m) + Bs}{s(s + c/m)} \qquad (2)$$

By equating like terms in the numerator, the following must hold

$$g = A\frac{c}{m} \qquad\qquad 0 = As + Bs$$

The first equation can be solved for $A = mg/c$. According to the second equation, $B = -A$, so $B = -mg/c$. Substituting these back into (2) gives

$$\frac{g}{s(s + c/m)} = \frac{mg/c}{s} - \frac{mg/c}{s + c/m}$$

This can be substituted into Eq. 1 to give

$$V(s) = \frac{v(0)}{s + c/m} + \frac{mg/c}{s} - \frac{mg/c}{s + c/m}$$

Taking inverse Laplace transforms yields

$$v(t) = v(0)e^{-(c/m)t} + \frac{mg}{c} - \frac{mg}{c}e^{-(c/m)t}$$

or collecting terms

$$v(t) = v(0)e^{-(c/m)t} + \frac{mg}{c}\left(1 - e^{-(c/m)t}\right)$$

1.2 Repeat Example 1.2. Compute the velocity to $t = 10$ s, with a step size of (a) 1 and (b) 0.5 s. Can you make any statement regarding the errors of the calculation based on the results?

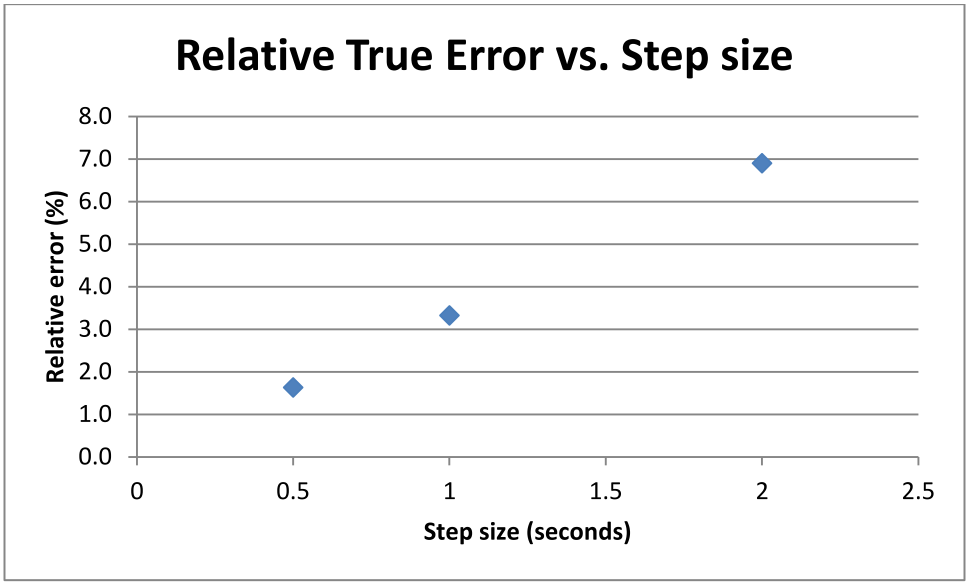
At $t = 10$ s, the analytical solution is 44.91893 (Example 1.1). The relative error can be calculated with

$$\text{relative true error} = \left|\frac{\text{analytical} - \text{numerical}}{\text{analytical}}\right| \times 100\%$$

The numerical results are:

step	v(10)	magnitude of relative error
2	48.0179	6.899%
1	46.4112	3.322%
0.5	45.6509	1.630%

The error versus step size can then be plotted as



Thus, halving the step size approximately halves the error.

1.3 Rather than the linear relationship of Eq. (1.7), you might choose to model the upward force on the parachutist as a second-order relationship,

$$F_U = -c' v |v|$$

where c' = a bulk second-order drag coefficient (kg/m). Note that the second-order term could be represented as v^2 if the parachutist always fell in the downward direction. For the present case, we use the more general representation, $v|v|$, so that the proper sign is obtained for both the downward and the upward directions.

(a) Using calculus, obtain the closed-form solution for the case where the jumper is initially at rest ($v = 0$ at $t = 0$).

(b) Repeat the numerical calculation in Example 1.2 with the same initial condition and parameter values, but with second-order drag. Use a value of 0.225 kg/m for c_d'.

(a) You are given the following differential equation with the initial condition, $v(t = 0) = 0$,

$$\frac{dv}{dt} = g - \frac{c'}{m}v^2$$

Multiply both sides by m/c' gives

$$\frac{m}{c'}\frac{dv}{dt} = \frac{m}{c'}g - v^2$$

Define $a = \sqrt{mg/c'}$

$$\frac{m}{c'}\frac{dv}{dt} = a^2 - v^2$$

Integrate by separation of variables,

$$\int \frac{dv}{a^2 - v^2} = \int \frac{c'}{m} dt$$

A table of integrals can be consulted to find that

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{a} \tanh^{-1} \frac{x}{a}$$

Therefore, the integration yields

$$\frac{1}{a} \tanh^{-1} \frac{v}{a} = \frac{c'}{m} t + C$$

If $v = 0$ at $t = 0$, then because $\tanh^{-1}(0) = 0$, the constant of integration $C = 0$ and we obtain the equation

$$\frac{1}{a} \tanh^{-1} \frac{v}{a} = \frac{c'}{m} t$$

This result can then be rearranged to solve for v

$$v = \sqrt{\frac{gm}{c'}} \tanh\left(\sqrt{\frac{gc'}{m}} t \right)$$

(b) Using Euler's method, the first two steps are computed

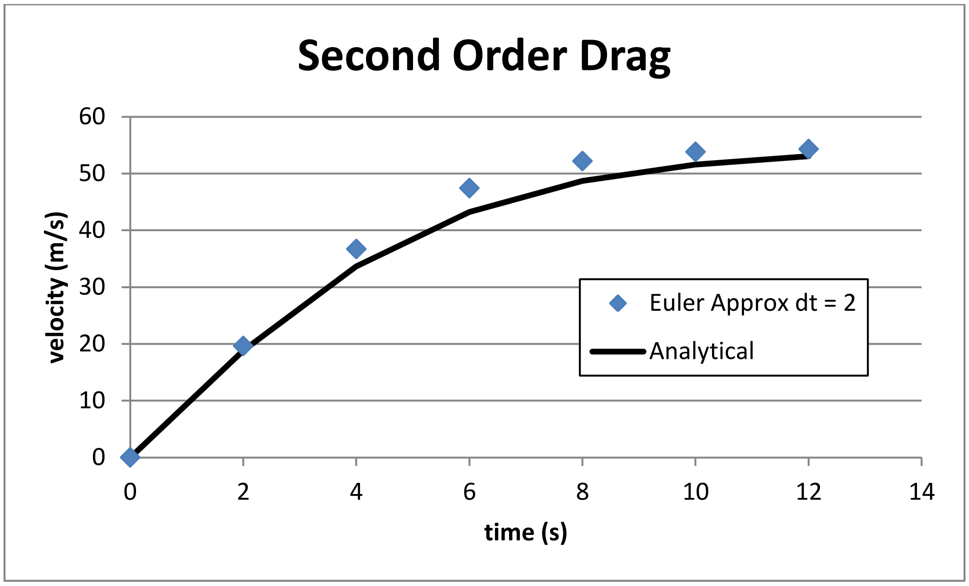
$$v(2) = 0 + \left[9.81 - \frac{0.225}{68.1} (0)^2 \right] 2 = 19.62$$

$$v(4) = 19.62 + \left[9.81 - \frac{0.225}{68.1} (19.62)^2 \right] 2 = 36.69631454$$

The computation can be continued and the results summarized along with the analytical result as:

t	***v-numerical***	***dv/dt***	***v-analytical***
0	0	9.81	0
2	19.62	8.538157	18.8138836
4	36.69631454	5.360817	33.61984724
6	47.41794779	2.381162	43.22542283
8	52.18027088	0.814029	48.7004867
10	53.80832813	0.243911	51.59332241
12	54.29615076	0.069674	53.06072073
∞	54.48999908	0	54.48999908

A plot of the numerical and analytical results can be developed



1.4 For the free-falling parachutist with linear drag, assume a first jumper is 70 kg and has a drag coefficient of 12 kg/s. If a second jumper has a drag coefficient of 15 kg/s and a mass of 75 kg, how long will it take him to reach the same velocity the first jumper reached in 10 s?

$$v(t) = \frac{gm}{c}(1 - e^{-(c/m)t})$$

Solve the equation for time as a function of velocity

$$t = -\frac{m}{c}\ln\left(1 - \frac{vc}{mg}\right)$$

jumper #1: $v(t) = \dfrac{9.81(70)}{12}(1 - e^{-(12/70)\,9}) = 46.91922$

jumper #2: $46.91922 = \dfrac{9.81(75)}{15}(1 - e^{-(15/75)\,t})$

$$t = -\frac{75}{15}\ln\left(1 - \frac{15(46.91922)}{75(9.81)}\right) = 15.68175$$

The second jumper will reach the t=10 s velocity of the first jumper after about 15.68 seconds.

1.5 Compute the velocity of a parachutist using Euler's method for the case where m = 80 kg and c = 10 kg/s. Perform the calculation from t = 0 to 20 s with a step size of 1 s. Use an initial condition that the parachutist has an upward velocity of 20 m/s at t = 0. At t = 10 s, assume that the chute is instantaneously deployed so that the drag coefficient jumps to 50 kg/s.

Before the chute opens (t < 10), Euler's method can be implemented as

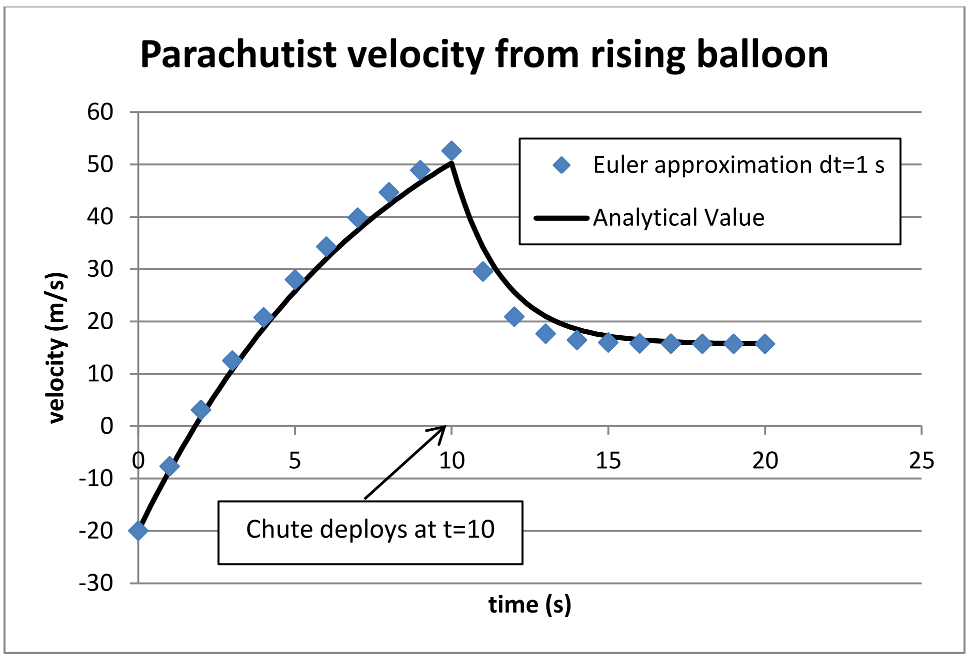
$$v(t + \Delta t) = v(t) + \left[9.81 - \frac{10}{80}v(t)\right]\Delta t$$

After the chute opens ($t \geq 10$), the drag coefficient is changed and the implementation becomes

$$v(t+\Delta t) = v(t) + \left[9.81 - \frac{50}{80}v(t)\right]\Delta t$$

Here is a summary of the results along with a plot:

Chute closed			Chute opened		
t	***v***	***dv/dt***	***t***	***v***	***dv/dt***
0	-20.00000	12.31000	10	52.57232	-23.04770
1	-7.69000	10.77125	11	29.52462	-8.64289
2	3.08125	9.42484	12	20.88173	-3.24108
3	12.50609	8.24674	13	17.64065	-1.21541
4	20.75283	7.21590	14	16.42524	-0.45578
5	27.96873	6.31391	15	15.96947	-0.17092
6	34.28264	5.52467	16	15.79855	-0.06409
7	39.80731	4.83409	17	15.73446	-0.02404
8	44.64139	4.22983	18	15.71042	-0.00901
9	48.87122	3.70110	19	15.70141	-0.00338
			20	15.69803	-0.00127



1.6 The following information is available for a bank account:

Date	Deposits	Withdrawals	Interest	Balance
5/1				1512.33
	220.13	327.26		
6/1				
	216.80	378.61		
7/1				
	450.25	106.80		
8/1				
	127.31	350.61		
9/1				

Note that the money earns interest, which is computed as

$$\text{Interest} = iB_i$$

where i = the interest rate expressed as a fraction per month and B_i = the initial balance at the beginning of the month.

(a) Use the conservation of cash to compute the balance on 6/1, 7/1, 8/1, and 9/1 if the interest rate is 1% per month (i = 0.01/month). Show each step in the computation.

(b) Write a differential equation for the cash balance in the form

$$\frac{dB}{dt} = f(D(t), W(t), i)$$

where t = time (months), $D(t)$ = deposits as a function of time ($/month), $W(t)$ = withdrawals as a function of time ($/month). For this case, assume that interest is compounded continuously; that is, interest = iB.

(c) Use Euler's method with a time step of 0.5 month to simulate the balance. Assume that the deposits and withdrawals are applied uniformly over the month.

(d) Develop a plot of balance versus time for **(a)** and **(c)**.

(a) This is a transient computation. For the period ending June 1:

Balance = Previous Balance + Deposits – Withdrawals + Interest
Balance = 1522.33 + 220.13 – 327.26 + 0.01(1522.33) = 1430.42

Note that the interest added to the account is for maintaining the account the previous month; for example, the interest collected during from May 1 to June 1 is calculated on the May 1st balance.
The balances for the remainder of the periods can be computed in a similar fashion as tabulated below:

Date	Deposit	Withdrawal	Interest	Balance
1-May				$ 1,512.33
	$ 220.13	$ 327.26	$ 15.12	
1-Jun				$ 1,420.32
	$ 216.80	$ 378.61	$ 14.20	
1-Jul				$ 1,272.72
	$ 450.25	$ 106.80	$ 12.73	
1-Aug				$ 1,628.89
	$ 127.31	$ 350.61	$ 16.29	
1-Sep				$ 1,421.88

(b) $\frac{dB}{dt} = D(t) - W(t) + iB$

(c) for t = 0 to 0.5:

$$\frac{dB}{dt} = 220.13 - 327.26 + 0.01(1512.33) = -92.01$$

$$B(0.5) = 1512.33 - 92.01(0.5) = 1466.33$$

for t = 0.5 to 1:

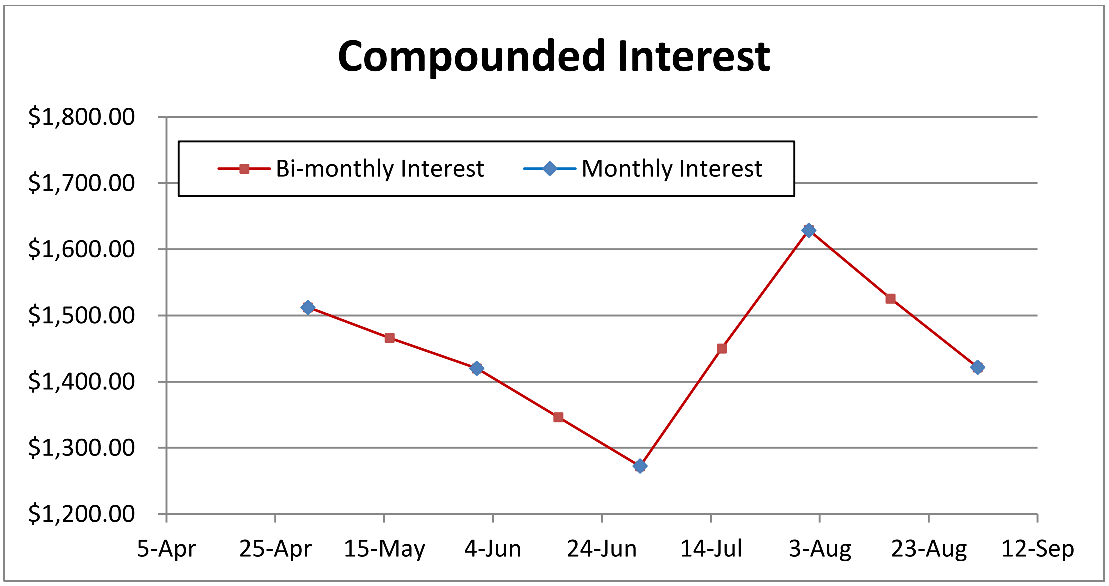
$$\frac{dB}{dt} = 220.13 - 327.260 + 0.01(1466.33) = -92.47$$

$$B(0.5) = 1466.33 - 92.47(0.5) = 1420.09$$

The balances for the remainder of the periods can be computed in a similar fashion as tabulated below. Parenthesis indicate negative numbers.

Date	Deposit	Withdrawal	Interest	*dB*/*dt*	Balance
1-May	\$ 220.13	\$ 327.26	\$ 15.12	\$ (92.01)	\$ 1,512.33
16-May	\$ 220.13	\$ 327.26	\$ 14.66	\$ (92.47)	\$ 1,466.33
1-Jun	\$ 216.80	\$ 378.61	\$ 14.20	\$ (147.61)	\$ 1,420.09
16-Jun	\$ 216.80	\$ 378.61	\$ 13.46	\$ (148.35)	\$ 1,346.29
1-Jul	\$ 450.25	\$ 106.80	\$ 12.72	\$ 356.17	\$ 1,272.12
16-Jul	\$ 450.25	\$ 106.80	\$ 14.50	\$ 357.95	\$ 1,450.20
1-Aug	\$ 127.31	\$ 350.61	\$ 16.29	\$ (207.01)	\$ 1,629.18
16-Aug	\$ 127.31	\$ 350.61	\$ 15.26	\$ (208.04)	\$ 1,525.67
1-Sep					\$ 1,421.65

(*d*) As in the plot below, the results of the two approaches are very close.



1.7 The amount of a uniformly distributed radioactive contaminant contained in a closed reactor is measured by its concentration c (becquerel/liter, or Bq/L). The contaminant decreases at a decay rate proportional to its concentration—that is,

$$\text{decay rate} = -kc$$

where k is a constant with units of day^{-1}. Therefore, according to Eq. (1.13), a mass balance for the reactor can be written as

$$\frac{dc}{dt} = -kc$$

$$\begin{pmatrix}\text{change}\\ \text{in mass}\end{pmatrix} = \begin{pmatrix}\text{decrease}\\ \text{by decay}\end{pmatrix}$$

(a) Use Euler's method to solve this equation from $t = 0$ to 1 d with $k = 0.2$ d^{-1}. Employ a step size of $\Delta t = 0.1$. The concentration at $t = 0$ is 10 Bq/L.

(b) Plot the solution on a semilog graph (i.e., ln c versus t) and determine the slope. Interpret your results.

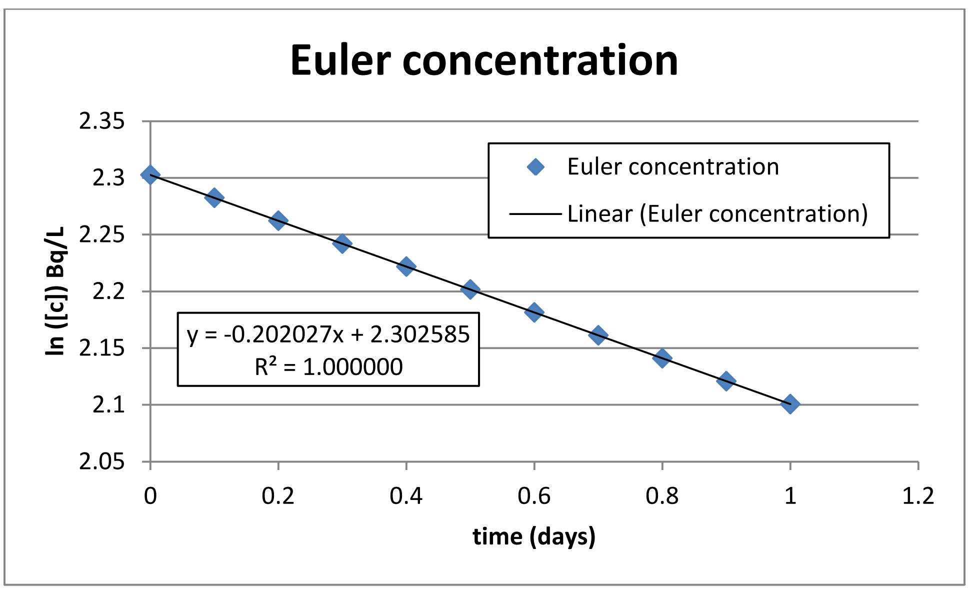
(a) The first two steps are

$$c(0.1) = 100 - 0.2(10)0.1 = 9.8 \text{ Bq/L}$$
$$c(0.2) = 9.80 - 0.2(9.80)0.1 = 9.604 \text{ Bq/L}$$

The process can be continued to yield

t	*c*	*dc*/*dt*
0	10.0000	-2.0000
0.1	9.8000	-1.9600
0.2	9.6040	-1.9208
0.3	9.4119	-1.8824
0.4	9.2237	-1.8447
0.5	9.0392	-1.8078
0.6	8.8584	-1.7717
0.7	8.6813	-1.7363
0.8	8.5076	-1.7015
0.9	8.3375	-1.6675
1	8.1707	-1.6341

(b) The results when plotted on a semi-log plot yields a straight line



The slope of this line can be estimated from the first and last points as

$$\frac{\ln(8.1707) - \ln(10)}{1} = -0.2020$$

Thus, the slope is approximately equal to the negative of the decay rate, within 10%. If we had used a smaller step size, the result would be more exact. A linear regression fit is shown on the graph using all points.

1.8 A group of 35 students attend a class in a room that measures 11 m by 8 m by 3 m. Each student takes up about 0.075 m^3 and gives out about 80 W of heat (1 W = 1 J/s). Calculate the air temperature rise during the first 15 minutes of the class if the room is completely sealed and insulated. Assume the heat capacity, C_υ, for air is 0.718 kJ/(kg K). Assume air is an ideal gas at 20°C and 101.325 kPa. Note that the heat absorbed by the air Q is related to the mass of the air m, the heat capacity, and the change in temperature by the following relationship:

$$Q = m\int_{T_1}^{T_2} C_\upsilon \, dT = mC_\upsilon (T_2 - T_1)$$

The mass of air can be obtained from the ideal gas law:

$$PV = \frac{m}{\text{Mwt}} RT$$

where P is the gas pressure, V is the volume of the gas, Mwt is the molecular weight of the gas (for air, 28.97 kg/kmol), and R is the ideal gas constant [8.314 kPa m^3/(kmol K)].

$$Q_{\text{students}} = 35 \text{ ind} \times 80 \frac{\text{J}}{\text{ind s}} \times 15 \text{ min} \times 60 \frac{\text{s}}{\text{min}} \times \frac{\text{kJ}}{1000 \text{ J}} = 2{,}520 \text{ kJ}$$

$$m = \frac{PV\text{Mwt}}{RT} = \frac{(101.325 \text{ kPa})(11\text{m} \times 8\text{m} \times 3\text{m} - 35 \times 0.075 \text{ m}^3)(28.97 \text{ kg/kmol})}{(8.314 \text{ kPa m}^3/(\text{kmol K}))((20 + 273.15)\text{K})} = 314.796 \text{ kg}$$

$$\Delta T = \frac{Q_{\text{students}}}{mC_v} = \frac{2{,}520 \text{ kJ}}{(314.796 \text{ kg})(0.718 \text{ kJ/(kg K)})} = 11.14928 \text{ K}$$

The final temperature is 20 + 11.14928 = 31.14928 °C**,** though only ΔT is requested.

1.9 A storage tank contains a liquid at depth y, where $y = 0$ when the tank is half full (Fig. P1.9). Liquid is withdrawn at a constant flow rate Q to meet demands. The contents are resupplied at a sinusoidal rate $3Q \sin^2(t)$.

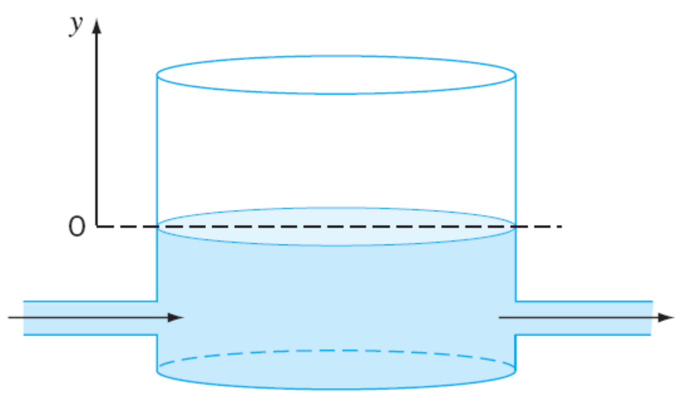


FIGURE P1.9

Equation (1.13) can be written for this system as

$$\frac{d(Ay)}{dt} = 3Q\sin^2(t) - Q$$

$$\left(\begin{array}{c}\text{change in} \\ \text{volume}\end{array}\right) = (\text{inflow}) - (\text{outflow})$$

or, since the surface area A is constant,

$$\frac{dy}{dt} = 3\frac{Q}{A}\sin^2(t) - \frac{Q}{A}$$

Use Euler's method to solve for the depth y from $t = 0$ to 10 d with a step size of 0.5 d. The parameter values are $A = 1200$ m^2 and $Q = 500$ m^3/d. Assume that the initial condition is $y = 0$.

The first two steps yield

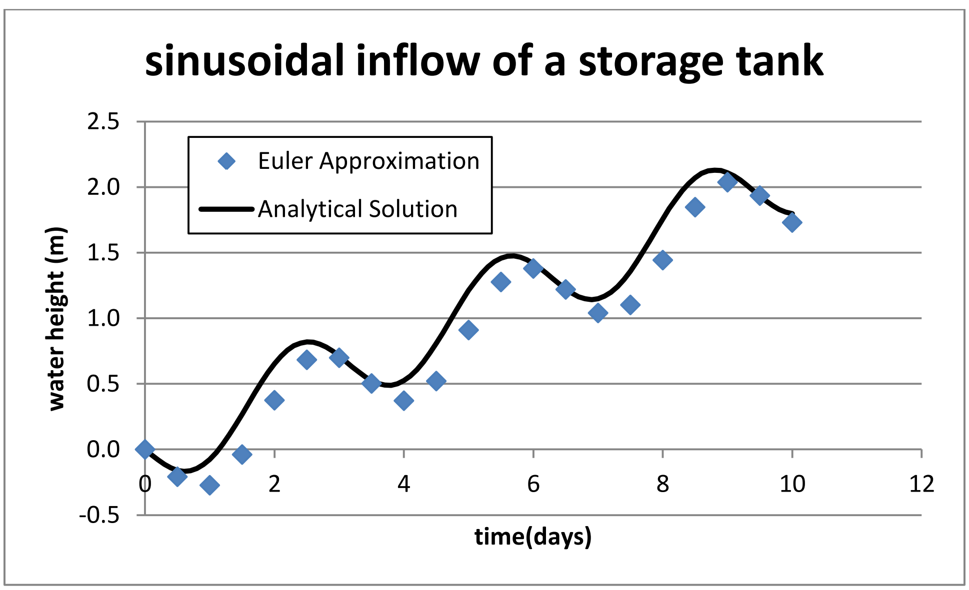
$$y(0.5) = 0 + \left[3\frac{500}{1200}\sin^2(0) - \frac{500}{1200}\right]0.5 = \; 0 + (-0.41667)\; 0.5 = -0.20833$$

$$y(1) = -0.20833 + \left[3\frac{500}{1200}\sin^2(0.5) - \frac{500}{1200}\right]0.5 = \; -0.20833 + (-0.12936)\; 0.5 = -0.23588$$

The process can be continued to give the following table and plot:

t	*y*	*dy*/*dt*	*t*	*y*	*dy*/*dt*
0	0.00000	-0.41667	5.5	1.27629	0.20557
0.5	-0.20833	-0.12936	6	1.37907	-0.31908
1	-0.27301	0.46843	6.5	1.21953	-0.35882
1.5	-0.03880	0.82708	7	1.04012	0.12287
2	0.37474	0.61686	7.5	1.10156	0.68314
2.5	0.68317	0.03104	8	1.44313	0.80687
3	0.69869	-0.39177	8.5	1.84656	0.38031
3.5	0.50281	-0.26286	9	2.03672	-0.20436
4	0.37138	0.29927	9.5	1.93453	-0.40961
4.5	0.52101	0.77779	10	1.72973	-0.04672
5	0.90991	0.73275			

graph of result of Euler. Analytical solution also shown as a cross-check. Smaller step sizes match better.



1.10 For the same storage tank described in Prob. 1.9, suppose that the outflow is not constant but rather depends on the depth. For this case, the differential equation for depth can be written as

$$\frac{dy}{dt} = 3\frac{Q}{A}\sin^2(t) - \frac{a(1+y)^{1.5}}{A}$$

Use Euler's method to solve for the depth y from $t = 0$ to 10 d with a step size of 0.5 d. The parameter values are $A = 1200$ m^2, $Q = 500$ m^3/d, and $\alpha = 150$. Assume that the initial condition is $y = 0$.

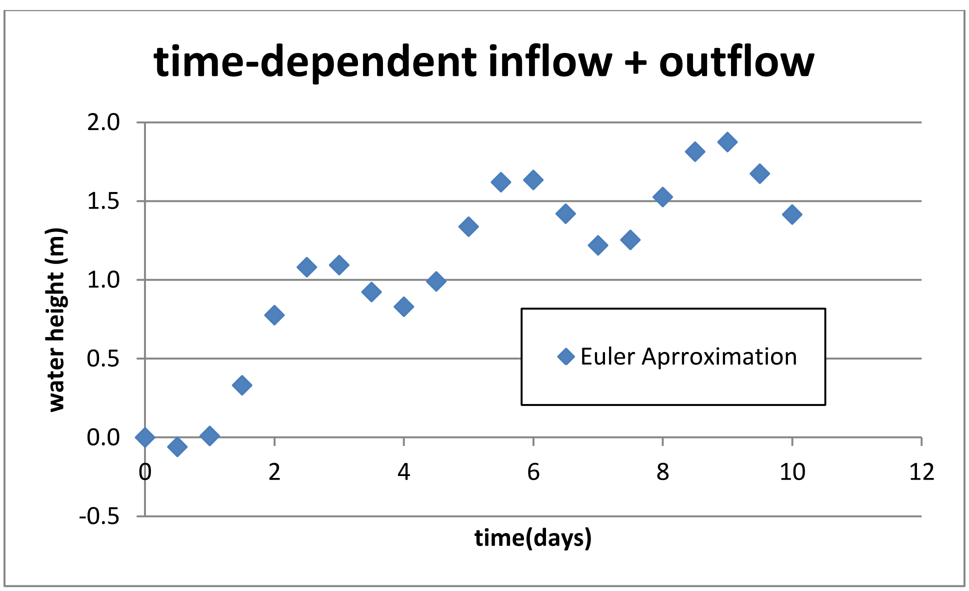
The first two steps yield

$$y(0.5) = 0 + \left[3\frac{500}{1200}\sin^2(0) - \frac{150(1+0)^{1.5}}{1200}\right]0.5 = 0 - 0.12(0.5) = \; -0.0625$$

$$y(1) = -0.0625 + \left[3\frac{500}{1200}\sin^2(0.5) - \frac{150(1-0.0625)^{1.5}}{1200}\right]0.5 = -0.0625 + 0.17384(0.5) = 0.02442$$

The process can be continued to give

t	*y*	*dy/dt*	*t*	*y*	*dy/dt*
0	0.00000	-0.12500	5.5	1.87262	0.01364
0.5	-0.06250	0.17384	6	1.87944	-0.51317
1	0.02442	0.75548	6.5	1.62286	-0.47313
1.5	0.40216	1.03620	7	1.38629	0.07876
2	0.92027	0.70090	7.5	1.42567	0.62757
2.5	1.27072	0.02000	8	1.73946	0.65677
3	1.28072	-0.40565	8.5	2.06784	0.12530
3.5	1.07789	-0.22060	9	2.13049	-0.48005
4	0.96759	0.37094	9.5	1.89046	-0.60721
4.5	1.15306	0.79955	10	1.58686	-0.15013
5	1.55284	0.63957			



1.11 Apply the conservation of volume (see Prob. 1.9) to simulate the level of liquid in a conical storage tank (Fig. P1.11). The liquid flows in at a sinusoidal rate of $Q_{in} = 3\sin^2(t)$ and flows out according to

$$Q_{out} = 3(y - y_{out})^{1.5} \qquad y > y_{out}$$
$$Q_{out} = 0 \qquad y \le y_{out}$$

where flow has units of m^3/d and y = the elevation of the water surface above the bottom of the tank (m). Use Euler's method to solve for the depth y from $t = 0$ to 8 d with a step size of 0.4 d. The parameter values are $r_{top} = 2.5$ m, $y_{top} = 4$ m, and $y_{out} = 1$ m. Assume that the level is initially below the outlet pipe with $y(0) = 0.75$ m.

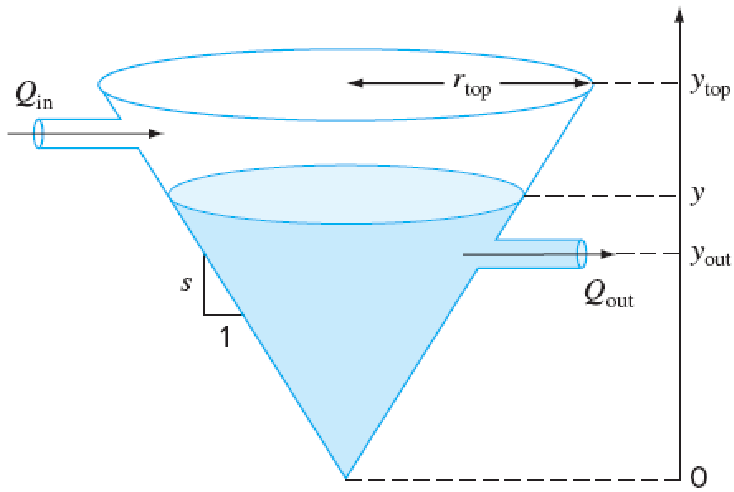


FIGURE P1.11

When the water level is above the outlet pipe, the volume balance can be written as

$$\frac{dV}{dt} = 3\sin^2(t) - 3(y - y_{out})^{1.5}$$

In order to solve this equation, we must relate the volume to the level. To do this, we recognize that the volume of a cone is given by $V = \pi r^2 y/3$. Defining the side slope as $s = y_{top}/r_{top}$, the radius can be related to the level ($r = y/s$) and the volume can be reexpressed as

$$V = \frac{\pi}{3s^2} y^3$$

which can be solved for

$$y = \sqrt[3]{\frac{3s^2 V}{\pi}} \qquad (1)$$

and substituted into the volume balance

$$\frac{dV}{dt} = 3\sin^2(t) - 3\left(\sqrt[3]{\frac{3s^2V}{\pi}} - y_{\text{out}}\right)^{1.5} \tag{2}$$

For the case where the level is below the outlet pipe, outflow is zero and the volume balance simplifies to

$$\frac{dV}{dt} = 3\sin^2(t) \tag{3}$$

These equations can then be used to solve the problem. Using the side slope of $s = 4/2.5 = 1.6$, the initial volume can be computed as

$$V(0) = \frac{\pi}{3(1.6)^2}0.75^3 = 0.17257 \text{ m}^3$$

For the first step, $y < y_{\text{out}}$ and Eq. (3) gives

$$\frac{dV}{dt}(0) = 3\sin^2(0) = 0$$

and Euler's method yields

$$V(0.4) = V(0) + \frac{dV}{dt}(0)\Delta t = 0.17257 + 0(0.4) = 0.17257$$

For the second step, Eq. (3) still holds and

$$\frac{dV}{dt}(0.4) = 3\sin^2(0.4) = 0.45494$$

$$V(0.8) = V(0.4) + \frac{dV}{dt}(0.4)\Delta t = 0.17257 + 0.45494(0.4) = 0.35455$$

Equation (1) can then be used to compute the new level, which is still below the output pipe. For the thirs step, V(1.2) = 0.97207, and

$$y = \sqrt[3]{\frac{3(1.6)^2(0.97207)}{\pi}} = 1.33445 \text{ m}$$

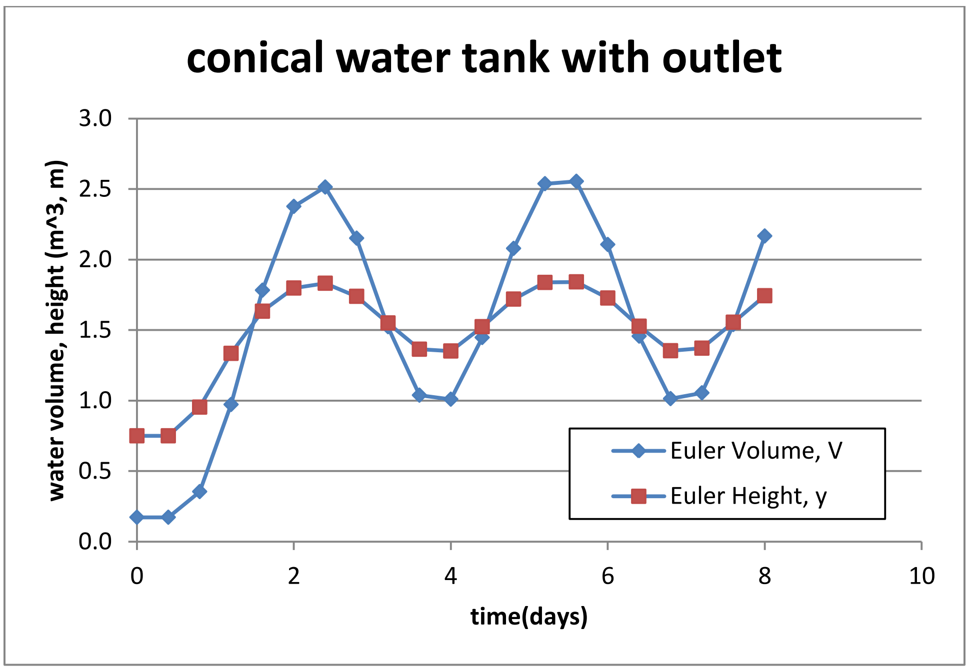
Because this level is now higher than the outlet pipe, Eq. (2) holds for the next step

$$\frac{dV}{dt}(1.2) = 1.543799 - 3\left(1.33445 - 1\right)^{1.5} = 2.025833$$

$$V(1.6) = 0.97207 + 2.025833(0.4) = 1.78240$$

The remainder of the calculation is summarized in the following table and figure.

t	Q_{in}	*V*	*y*	Q_{out}	*dV/dt*
0	0	0.17257	0.75000	0	0
0.4	0.45494	0.17257	0.75000	0	0.45494
0.8	1.543799	0.35455	0.95345	0	1.543799
1.2	2.606091	0.97207	1.33445	0.580257	2.025833
1.6	2.997442	1.78240	1.63332	1.512029	1.485413
2	2.480465	2.37657	1.79771	2.137429	0.343037
2.4	1.368752	2.51378	1.83167	2.275332	-0.90658
2.8	0.336651	2.15115	1.73898	1.90576	-1.56911
3.2	0.010223	1.52351	1.55007	1.223913	-1.21369
3.6	0.587473	1.03803	1.36398	0.658768	-0.07129
4	1.71825	1.00951	1.35137	0.624839	1.093411
4.4	2.71664	1.44688	1.52363	1.136748	1.579892
4.8	2.977032	2.07883	1.71927	1.830027	1.147004
5.2	2.341476	2.53763	1.83744	2.299073	0.042403
5.6	1.195493	2.55460	1.84153	2.315915	-1.12042
6	0.234219	2.10643	1.72684	1.859008	-1.62479
6.4	0.040751	1.45651	1.52701	1.147755	-1.107
6.8	0.732444	1.01371	1.35324	0.629836	0.102608
7.2	1.889726	1.05475	1.37126	0.678647	1.211079
7.6	2.810605	1.53919	1.55537	1.241642	1.568964
8	2.936489	2.16677	1.74318	1.922027	1.014462



1.12 In our example of the free-falling parachutist, we assumed that the acceleration due to gravity was a constant value. Although this is a decent approximation when we are examining falling objects near the surface of the earth, the gravitational force decreases as we move above sea level. A more general representation based on *Newton's inverse square law* of gravitational attraction can be written as

$$g(x) = g(0)\frac{R^2}{(R+x)^2}$$

where $g(x)$ = gravitational acceleration at altitude x (in m) measured upward from the earth's surface (m/s^2), $g(0)$ = gravitational acceleration at the earth's surface (≅ 9.81 m/s^2), and R = the earth's radius (@ 6.37´ 10^6 m).

(a) In a fashion similar to the derivation of Eq. (1.9), use a force balance to derive a differential equation for velocity as a function of time that utilizes this more complete representation of gravitation. However, for this derivation, assume that upward velocity is positive.
(b) For the case where drag is negligible, use the chain rule to express the differential equation as a function of altitude rather than time. Recall that the chain rule is

$$\frac{d\upsilon}{dt} = \frac{d\upsilon}{dx}\frac{dx}{dt}$$

(c) Use calculus to obtain the closed-form solution where $\upsilon = \upsilon_0$ at $x = 0$.
(d) Use Euler's method to obtain a numerical solution from $x = 0$ to 100,000 m using a step size of 10,000 m where the initial velocity is 1400 m/s upward. Compare your result with the analytical solution.

(a) The force balance can be written as:

$$m\frac{dv}{dt} = -mg(0)\frac{R^2}{(R+x)^2} + c_d v|v|$$

Dividing by mass gives

$$\frac{dv}{dt} = -g(0)\frac{R^2}{(R+x)^2} + \frac{c_d}{m}v|v|$$

(b) Recognizing that $dx/dt = v$, the chain rule is

$$\frac{dv}{dt} = v\frac{dv}{dx}$$

Setting drag to zero and substituting this relationship into the force balance gives

$$\frac{dv}{dx} = -\frac{g(0)}{v}\frac{R^2}{(R+x)^2}$$

(c) Using separation of variables

$$v\,dv = -g(0)\frac{R^2}{(R+x)^2}dx$$

Integrating gives

$$\frac{v^2}{2} = g(0)\frac{R^2}{R+x} + C$$

Applying the initial condition yields

$$\frac{v_0^2}{2} = g(0)\frac{R^2}{R+0} + C$$

which can be solved for $C = v_0^2/2 - g(0)R$, which can be substituted back into the solution to give

$$\frac{v^2}{2} = g(0)\frac{R^2}{R+x} + \frac{v_0^2}{2} - g(0)R$$

or

$$v = \pm\sqrt{v_0^2 + 2g(0)\frac{R^2}{R+x} - 2g(0)R}$$

Note that the plus sign holds when the object is moving upwards and the minus sign holds when it is falling.

(d) Euler's method can be developed as

$$v(x_{i+1}) = v(x_i) + \left[-\frac{g(0)}{v(x_i)}\frac{R^2}{(R+x_i)^2}\right](x_{i+1} - x_i)$$

The first step can be computed as

$$v(10,000) = 1,400 + \left[-\frac{9.81}{1,400}\frac{(6.37\times10^6)^2}{(6.37\times10^6+0)^2}\right](10,000-0) = 1,400 + (-0.00701)10,000 = 1329.9286$$

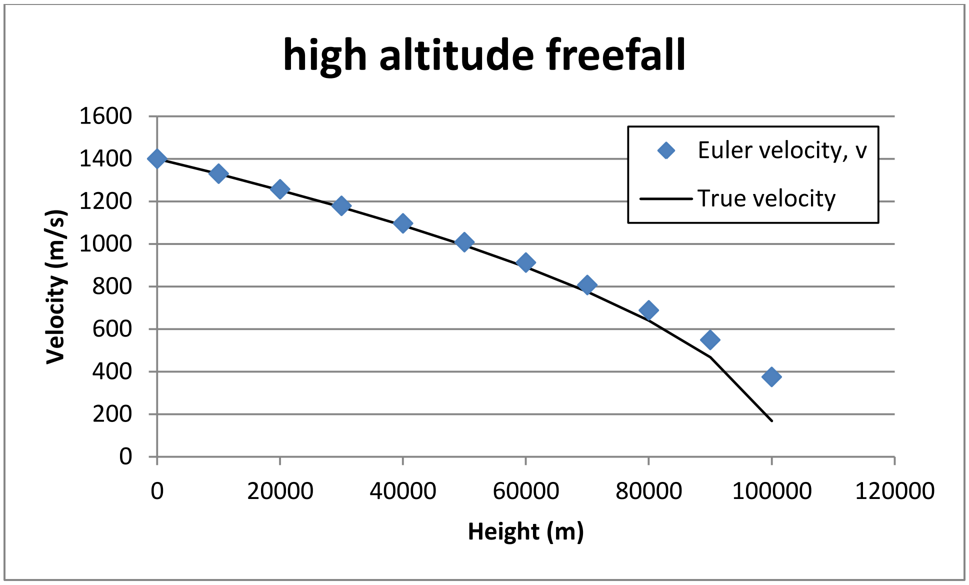
The remainder of the calculations can be implemented in a similar fashion as in the following table

x	*v*	*dv/dx*	*v*-analytical
0	1400	-0.00701	1400
10000	1329.9286	-0.00735	1328.197
20000	1256.3963	-0.00776	1252.529
30000	1178.8038	-0.00824	1172.245
40000	1096.3622	-0.00884	1086.323
50000	1007.9977	-0.00958	993.2976
60000	912.18608	-0.01055	890.946
70000	806.63991	-0.0119	775.5825
80000	687.65376	-0.01391	640.2093
90000	548.51163	-0.01739	467.7616
100000	374.61267	-0.02538	168.2991

For the analytical solution, the value at 10,000 m can be computed as

$$v = \sqrt{1,400^2 + 2(9.81)\frac{(6.37\times10^6)^2}{(6.37\times10^6+10,000)} - 2(9.81)(6.37\times10^6)} = 1328.197$$

The remainder of the analytical values can be implemented in a similar fashion as in the last column of the above table. The numerical and analytical solutions can be displayed graphically.



1.13 Suppose that a spherical droplet of liquid evaporates at a rate that is proportional to its surface area,

$$\frac{dV}{dt} = -kA$$

where *V* = volume (mm^3), *t* = time (min), *k* = the evaporation rate (mm/min), and *A* = surface area (mm^2). Use Euler's method to compute the volume of the droplet from *t* = 0 to 10 min using a step size of 0.25 min. Assume that *k* = 0.1 mm/min and that the droplet initially has a radius of 3 mm. Assess the validity of your results by determining the radius of your final computed volume and verifying that it is consistent with the evaporation rate.

The volume of the droplet is related to the radius as

$$V = \frac{4\pi r^3}{3} \tag{1}$$

This equation can be solved for radius as

$$r = \sqrt[3]{\frac{3V}{4\pi}} \tag{2}$$

The surface area is

$$A = 4\pi r^2 \tag{3}$$

Equation (2) can be substituted into Eq. (3) to express area as a function of volume

$$A = 4\pi \left(\frac{3V}{4\pi}\right)^{2/3}$$

This result can then be substituted into the original differential equation,

$$\frac{dV}{dt} = -k4\pi \left(\frac{3V}{4\pi}\right)^{2/3} \tag{4}$$

The initial volume can be computed with Eq. (1),

$$V = \frac{4\pi r^3}{3} = \frac{4\pi (3.0)^3}{3} = 113.0973 \text{ mm}^3$$

Euler's method can be used to integrate Eq. (4). For the first step, the result is

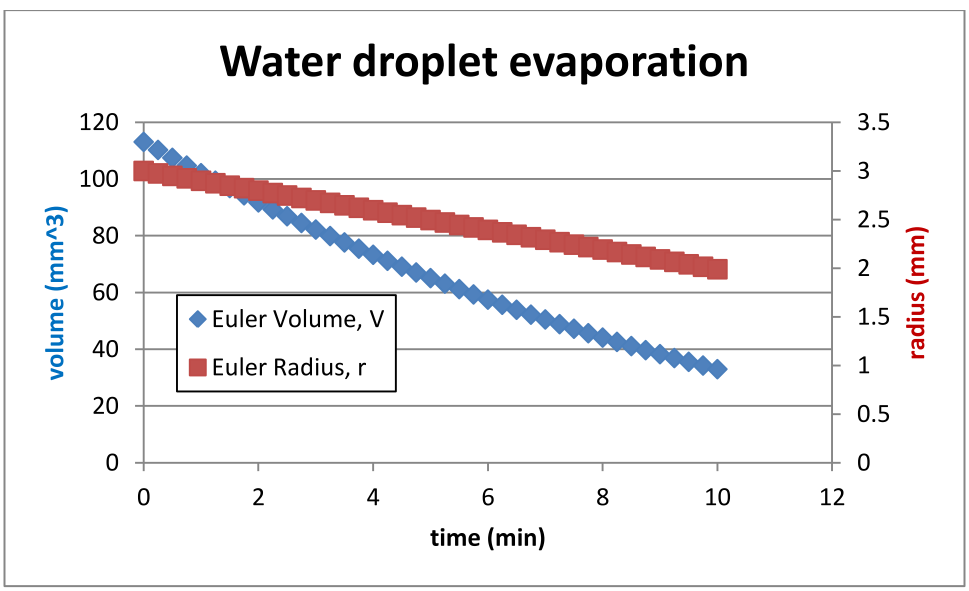
$$V(0.25) = V(0) + \frac{dV}{dt}(0) \times \Delta t = 113.0973 - 0.1(4)\pi\left(\frac{3(113.0973)}{4\pi}\right)^{2/3} \times 0.25$$

$$= 113.0973 - 11.3097(0.25) = 110.2699$$

Here are the beginning and ending steps

t	*V*	*dV/dt*
0	113.0973	-11.3097
0.25	110.2699	-11.1204
0.5	107.4898	-10.9327
0.75	104.7566	-10.7466
1	102.07	-10.5621
•		
•		
•		
9	38.29357	-5.49416
9.25	36.92003	-5.36198
9.5	35.57954	-5.2314
9.75	34.27169	-5.1024
10	32.99609	-4.97499

A plot of the results is shown below. We have included the radius on this plot (dashed line and right scale):



Eq. (2) can be used to compute the final radius as

$$r = \sqrt[3]{\frac{3(32.99609)}{4\pi}} = 1.989717$$

Therefore, the average evaporation rate can be computed as

$$k = \frac{(3.0 - 1.989717)\ \text{mm}}{10\ \text{min}} = 0.101028313 \frac{\text{mm}}{\text{min}}$$

which is approximately equal to the given evaporation rate of 0.10 mm/min.

1.14 *Newton's law of cooling* says that the temperature of a body changes at a rate proportional to the difference between its temperature and that of the surrounding medium (the ambient temperature),

$$\frac{dT}{dt} = -k(T - T_a)$$

where T = the temperature of the body (°C), t = time (min), k = the proportionality constant (per minute), and T_a = the ambient temperature (°C). Suppose that a cup of coffee originally has a temperature of 68°C. Use Euler's method to compute the temperature from t = 0 to 10 min using a step size of 1 min if T_a = 21°C and k = 0.019/min.

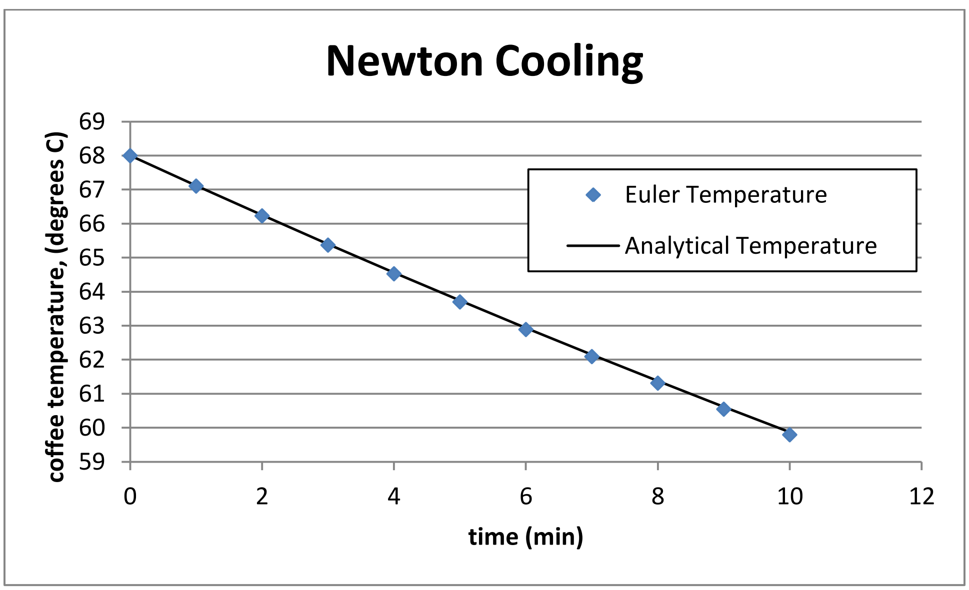
The first two steps can be computed as

$$T(1) = 68 + \left[-0.019(68 - 21)\right]2 = 68 + (-0.893)2 = 67.107$$

$$T(2) = 67.107 + \left[-0.019(67.107 - 21)\right]2 = 68.1 + (-0.87603)2 = 66.23097$$

The remaining results are displayed below along with a plot of the results.

t	*T*	*dT/dt*	*Tanalytical*	*t*	*T*	*dT/dt*	*Tanalytical*
0	68	-0.893	68	6	62.89015	-0.79591	62.93612
1	67.107	-0.87603	67.11543	7	62.09424	-0.78079	62.14686
2	66.23097	-0.85939	66.24751	8	61.31345	-0.76596	61.37245
3	65.37158	-0.84306	65.39592	9	60.54749	-0.7514	60.61261
4	64.52852	-0.82704	64.56036	10	59.79609	-0.73713	59.86708
5	63.70148	-0.81133	63.74053				



1.15 As depicted in Fig. P1.15, an *RLC circuit* consists of three elements: a resistor (R), an inductor (L), and a capacitor (C). The flow of current across each element induces a voltage drop. Kirchhoff's second voltage law states that the algebraic sum of these voltage drops around a closed circuit is zero,

$$iR + L\frac{di}{dt} + \frac{q}{C} = 0$$

where i = current, R = resistance, L = inductance, t = time, q = charge, and C = capacitance. In addition, the current is related to charge as in

$$\frac{dq}{dt} = i$$

(a) If the initial values are $i(0) = 0$ and $q(0) = 0.5$ C, use Euler's method to solve this pair of differential equations from $t = 0$ to 0.1 s using a step size of $\Delta t = 0.01$ s. Employ the following parameters for your calculation: $R = 250\ \Omega$, $L = 5$ H, and $C = 10^{-4}$ F.
(b) Develop a plot of i and q versus t.

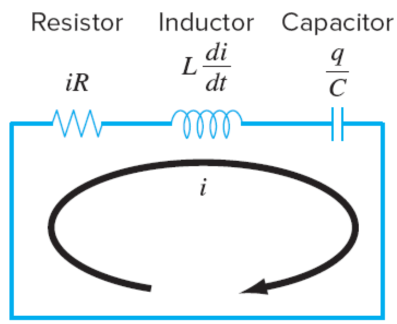


FIGURE P1.15

The pair of differential equations to be solved are

$$\frac{di}{dt} = -\frac{R}{L}i - \frac{1}{CL}q$$

$$\frac{dq}{dt} = i$$

or substituting the parameters

$$\frac{di}{dt} = -40i - 2{,}000q$$

$$\frac{dq}{dt} = i$$

The first step can be implemented by first using the differential equations to compute the slopes

$$\frac{di}{dt} = -40(0) - 2{,}000(1) = -2{,}000$$

$$\frac{dq}{dt} = 0$$

Then, Euler's method can be applied as

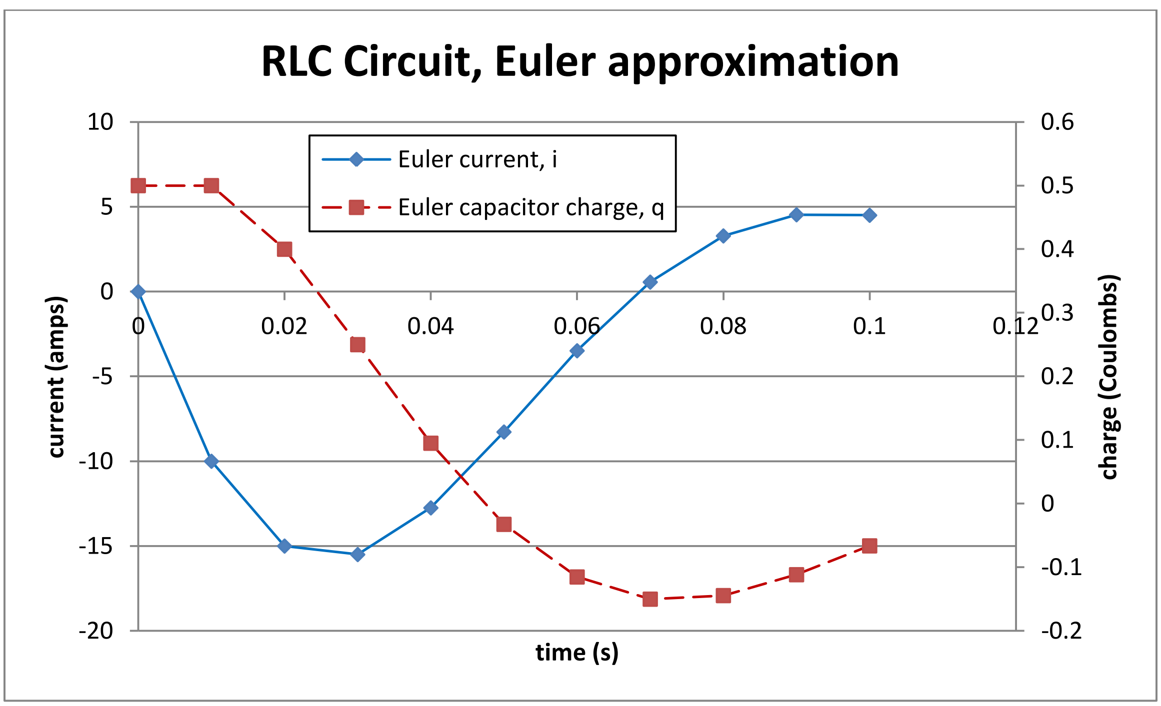
$$i(0.01) = 0 - 2,000(0.01) = -20$$
$$q(0.01) = 1 + 0(0.01) = 1$$

For the second step

$$\frac{di}{dt} = -40(-20) - 2,000(1) = -1,200$$
$$\frac{dq}{dt} = -20$$
$$i(0.02) = -20 - 1,200(0.01) = -32$$
$$q(0.02) = 1 - 20(0.01) = 0.8$$

The remaining steps are summarized in the following table and plot:

t	*i*	*q*	*di/dt*	*dq/dt*
0	0	0.5	-1000	0
0.01	-10	0.5	-500	-10
0.02	-15	0.4	-50	-15
0.03	-15.5	0.25	275	-15.5
0.04	-12.75	0.095	447.5	-12.75
0.05	-8.275	-0.0325	478.75	-8.275
0.06	-3.4875	-0.11525	404.875	-3.4875
0.07	0.56125	-0.15013	272.1875	0.56125
0.08	3.283125	-0.14451	124.8688	3.283125
0.09	4.5318125	-0.11168	-3.22813	4.531813
0.1	4.4995313	-0.06636	-92.2503	4.499531



1.16 A fluid is pumped into the network shown in Fig. P1.16. If $Q_2 = 0.7$, $Q_3 = 0.5$, $Q_7 = 0.1$, and $Q_8 = 0.3\ \text{m}^3/\text{s}$, determine the other flows.

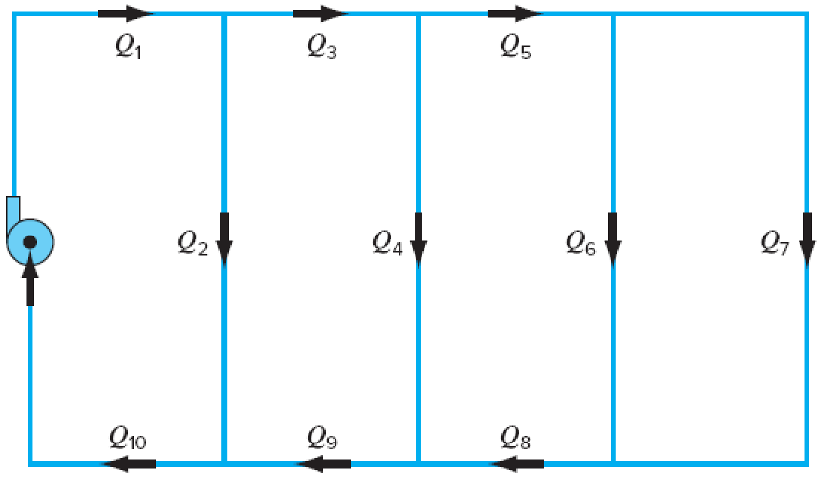
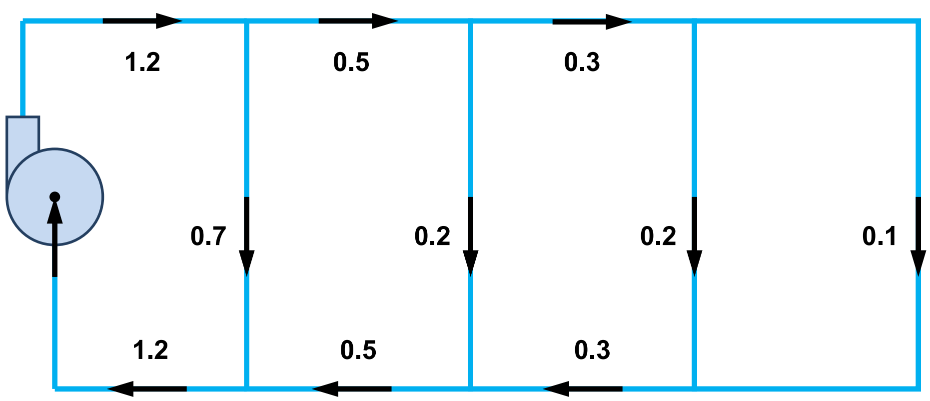


FIGURE P1.16

Continuity at the nodes can be used to determine the flows as follows:

$$Q_1 = Q_2 + Q_3 = 0.7 + 0.5 = 1.2\ \text{m}^3/\text{s}$$
$$Q_{10} = Q_1 = 1.2\ \text{m}^3/\text{s}$$
$$Q_9 = Q_{10} - Q_2 = 1.2 - 0.7 = 0.5\ \text{m}^3/\text{s}$$
$$Q_4 = Q_9 - Q_8 = 0.5 - 0.3 = 0.2\ \text{m}^3/\text{s}$$
$$Q_5 = Q_3 - Q_4 = 0.5 - 0.2 = 0.3\ \text{m}^3/\text{s}$$
$$Q_6 = Q_5 - Q_7 = 0.3 - 0.1 = 0.2\ \text{m}^3/\text{s}$$



1.17 The velocity is equal to the rate of change of distance x (m),

$$\frac{dx}{dt} = v(t) \qquad \text{(P1.17)}$$

(a) Substitute in Eq. (1.10) and develop an analytical solution for distance as a function of time. Assume that $x(0) = 0$.

(b) Use Euler's method to numerically integrate Eqs. (P1.17) and (1.9) in order to determine both the velocity and distance fallen as a function of time for the first 10 s of free-fall using the same parameters as in Example 1.2.
(c) Develop a plot of your numerical results together with the analytical solution.

(a) Substituting Eq. (1.10) into Eq. (P1.18) gives

$$\frac{dx}{dt} = \frac{gm}{c}(1 - e^{-(c/m)t})$$

Separation of variables gives

$$\int_0^x dx = \frac{gm}{c}\int_0^t 1 - e^{-(c/m)t}\,dt$$

Integration yields

$$x = \frac{gm}{c}t - \frac{gm^2}{c^2}(1 - e^{-(c/m)t})$$

(b) Euler's method can be applied for the first step as

$$\frac{dv}{dt}(0) = g - \frac{c}{m}v = 9.81 - \frac{12.5}{68.1}0 = 9.81$$

$$\frac{dx}{dt}(0) = v = 0$$

$$v(2) = v(0) + \frac{dv}{dt}(0)\Delta t = 0 + 9.81(2) = 19.62$$

$$x(2) = x(0) + \frac{dx}{dt}(0)\Delta t = 0 + 0(2) = 0$$

For the second step:

$$\frac{dv}{dt}(2) = 9.81 - \frac{12.5}{68.1}19.62 = 6.2087$$

$$\frac{dx}{dt}(0) = 19.62$$

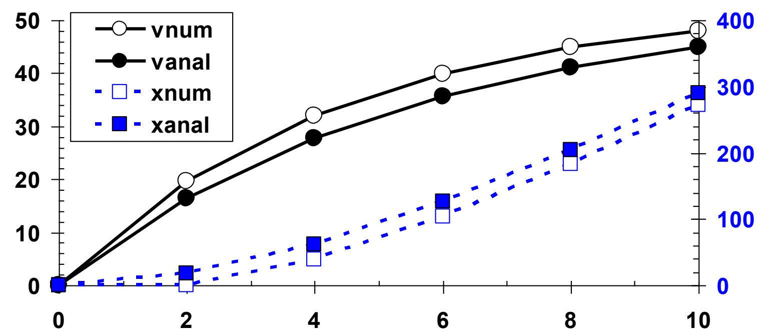
$$v(4) = 19.62 + 6.2087(2) = 32.0374$$

$$x(4) = 0 + 19.62(2) = 39.24$$

The remaining steps can be computed in a similar fashion as tabulated below along with the analytical solution:

t	*vnum*	*xnum*	*dv/dt*	*dx/dt*	*vanal*	*xanal*
0	0.0000	0.0000	9.8100	0.0000	0.0000	0.0000
2	19.6200	0.0000	6.2087	19.6200	16.4217	17.4242
4	32.0374	39.2400	3.9294	32.0374	27.7976	62.3380
6	39.8962	103.3147	2.4869	39.8962	35.6781	126.2949
8	44.8700	183.1071	1.5739	44.8700	41.1372	203.4435
10	48.0179	272.8472	0.9961	48.0179	44.9189	289.7305

(c)



1.18 You are working as a crime-scene investigator and must predict the temperature of a homicide victim over a 5-hr period. You know that the room where the victim was found was at 12°C when the body was discovered.

(a) Use *Newton's law of cooling* (Prob. 1.14) and Euler's method to compute the victim's body temperature for the 5-hr period using values of k = 0.12/hr and Δt = 0.5 hr. Assume that the victim's body temperature at the time of death was 37°C and that the room temperature was at a constant value of 12°C over the 5-hr period.

(b) Further investigation reveals that the room temperature had actually dropped linearly from 20 to 12°C over the 5-hr period. Repeat the same calculation as in **(a)** but incorporate this new information.

(c) Compare the results from **(a)** and **(b)** by plotting them on the same graph.

(a) For the constant temperature case, Newton's law of cooling is written as

$$\frac{dT}{dt} = -0.12(T - 10)$$

The first two steps of Euler's methods are

$$T(0.5) = T(0) - \frac{dT}{dt}(0) \times \Delta t = 37 + 0.12(12 - 37)(0.5) = 37 - 3 \times 0.50 = 35.5000$$

$$T(1) = 35.5000 + 0.12(12 - 35.5000)(0.5) = 35.3800 - 2.820 \times 0.50 = 34.0900$$

The remaining calculations are summarized in the following table:

t	***T_a***	***T***	***dT/dt***
0:00	12	37.0000	-3.0000
0:30	12	35.5000	-2.8200
1:00	12	34.0900	-2.6508
1:30	12	32.7646	-2.4918
2:00	12	31.5187	-2.3422
2:30	12	30.3476	-2.2017
3:00	12	29.2467	-2.0696
3:30	12	28.2119	-1.9454
4:00	12	27.2392	-1.8287
4:30	12	26.3249	-1.7190
5:00	12	25.4654	-1.6158

(b) For this case, the room temperature can be represented as

$$T_a = 20 - 1.6t$$

where t = time (hrs). Newton's law of cooling is written as

$$\frac{dT}{dt} = -0.12(T - 20 + 1.6t)$$

The first two steps of Euler's methods are

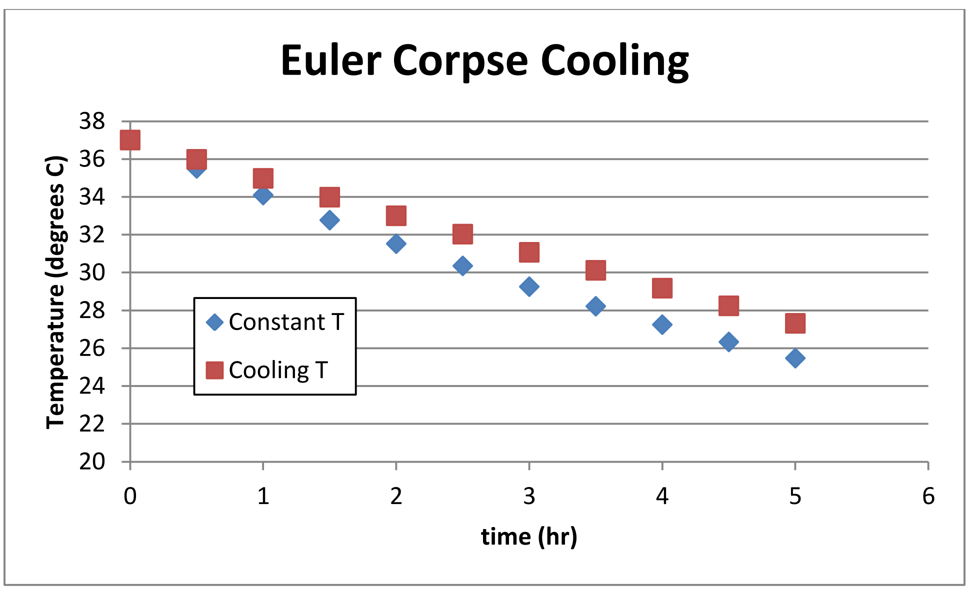
$$T(0.5) = 37 + 0.12(20 - 37)(0.5) = 37 - 2.040 \times 0.50 = 35.9800$$
$$T(1) = 35.9800 + 0.12(19.2 - 35.9800)(0.5) = 35.9800 - 2.0136 \times 0.50 = 34.9732$$

The remaining calculations are summarized in the following table:

t	T_a	*T*	*dT/dt*
0:00	20	37.0000	-2.0400
0:30	19.2	35.9800	-2.0136
1:00	18.4	34.9732	-1.9888
1:30	17.6	33.9788	-1.9655
2:00	16.8	32.9961	-1.9435
2:30	16	32.0243	-1.9229
3:00	15.2	31.0629	-1.9035
3:30	14.4	30.1111	-1.8853
4:00	13.6	29.1684	-1.8682
4:30	12.8	28.2343	-1.8521
5:00	12	27.3083	-1.8370

Comparison with (a) indicates that the effect of the room air temperature has a significant effect on the expected temperature at the end of the 5-hr period (difference = 27.3083 – 25.4654 = 1.8429°C).

(c) The solutions for (a) Constant T_a, and (b) Cooling T_a are plotted below:



1.19 Suppose that a parachutist with linear drag (m = 70 kg, c = 12.5 kg/s) jumps from an airplane flying at an altitude of 200 m with a horizontal velocity of 180 m/s relative to the ground.
(a) Write a system of four differential equations for x, y, $\upsilon_x = dx/dt$, and $\upsilon_y = dy/dt$.
(b) If the initial horizontal position is defined as x = 0, use Euler's methods with Δt = 1 s to compute the jumper's position over the first 10 s.
(c) Develop plots of y versus t and y versus x. Use the plots to graphically estimate when and where the jumper would hit the ground if the chute failed to open.
(d) At what angle would the parachutist be traveling in the last whole second before impact?

(a) $\dfrac{dx}{dt} = v_x \qquad \dfrac{dy}{dt} = v_y \qquad \dfrac{dv_x}{dt} = -\dfrac{c}{m}v_x \qquad \dfrac{dv_y}{dt} = -g + \dfrac{c}{m}\left|v_y\right|$

Note here that y is defined positive up, and resistance to gravity from air is thus dependent upon the magnitude of the velocity. Using the convention of y positive down removes the need for the absolute value, but results in a plot that needs to be inverted to provide a physical representation.

(b) The first step,

$$x(1) = x(0) + \frac{dx}{dt}\Delta t = 0 + 180(1) = 180$$

$$y(1) = y(0) + \frac{dy}{dt}\Delta t = -100 + 0(1) = -100$$

$$v_x(1) = v_x(0) + \frac{dv_x}{dt}\Delta t = 180 - \frac{12.5}{70}180(1) = 147.8571$$

$$v_y(1) = v_y(0) + \frac{dv_y}{dt}\Delta t = 0 + \left[9.81 - \frac{12.5}{70}(0)\right](1) = 9.81$$

The second step

$$x(2) = 180 + 147.8571(1) = 327.8571$$

$$y(1) = -100 + 9.81(1) = -90.19$$

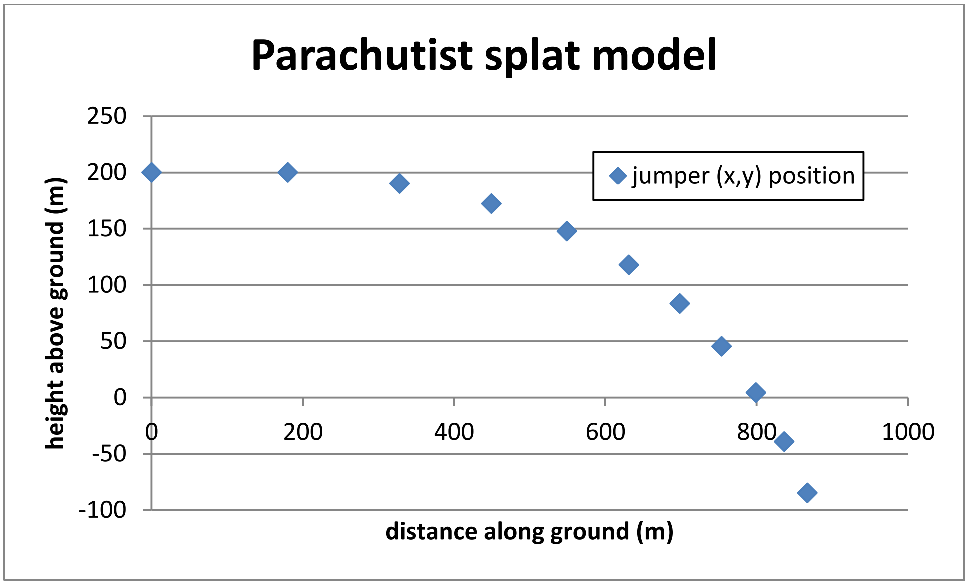
$$v_x(1) = 147.8571 - \frac{12.5}{70}147.8571(1) = 121.4541$$

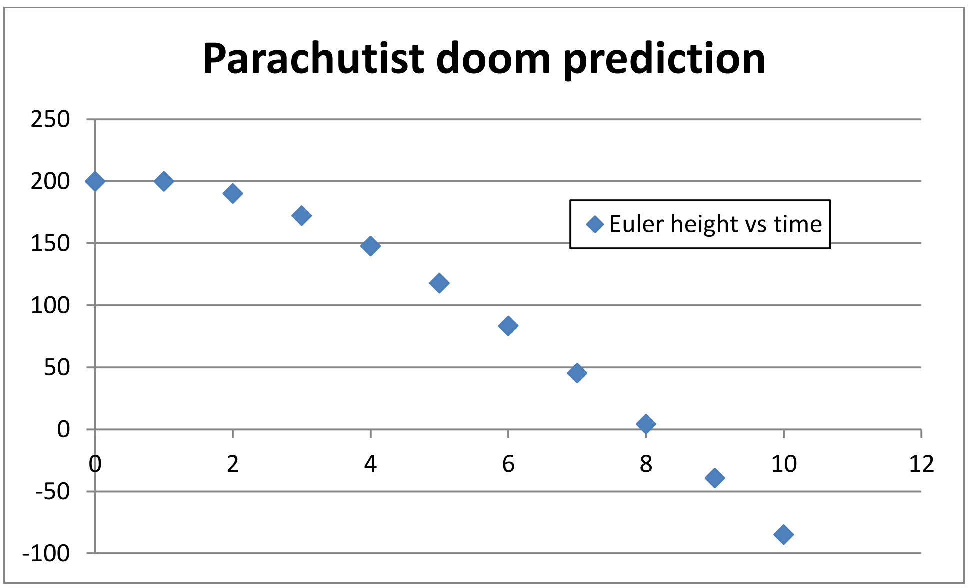
$$v_y(1) = 9.81 - \left[9.81 - \frac{12.5}{70}(9.81)\right](1) = 17.8682$$

These along with the remaining results can be tabulated as

t	x	y	v_x	v_y	dx/dt	dy/dt	dv_x/dt	dv_y/dt
0	0	200	180	0	180	0	-32.1429	-9.81
1	180	200	147.8571	-9.81	147.8571	-9.81	-26.4031	-8.05821
2	327.8571	190.19	121.4541	-17.8682	121.4541	-17.8682	-21.6882	-6.61925
3	449.3112	172.3218	99.76585	-24.4875	99.76585	-24.4875	-17.8153	-5.43724
4	549.0771	147.8343	81.95052	-29.9247	81.95052	-29.9247	-14.634	-4.4663
5	631.0276	117.9096	67.3165	-34.391	67.3165	-34.391	-12.0208	-3.66875
6	698.3441	83.51862	55.2957	-38.0598	55.2957	-38.0598	-9.87423	-3.01362
7	753.6398	45.45887	45.42147	-41.0734	45.42147	-41.0734	-8.11098	-2.47547
8	799.0613	4.385497	37.31049	-43.5488	37.31049	-43.5488	-6.66259	-2.03342
9	836.3718	-39.1633	30.6479	-45.5823	30.6479	-45.5823	-5.47284	-1.67031
10	867.0197	-84.7456	25.17506	-47.2526	25.17506	-47.2526	-4.49555	-1.37204

(c) The following plots indicates that the jumper will hit the ground in about t = 8 s at about x = 800 m.





(d) The angle relative to ground during the last whole second can be estimated from finite difference between seconds 7 and 8

$$\tan\theta = -\frac{y(8) - y(7)}{x(8) - x(7)}$$

$$\theta = \arctan\left(-\frac{y(8) - y(7)}{x(8) - x(7)}\right)$$

This angle can also be estimated as average angle between at $t = 7$ and 8 seconds from
$\tan\theta = -v_y(t)/v_x(t)$

Finite difference method: 0.735 radians, or 42 degrees
Average velocity angle method: (0.862 + 0.735)/2 = 0.80 radians, or 46 degrees

1.20 As noted in Prob. 1.3, drag is more accurately represented as depending on the square of velocity. A more fundamental representation of the drag force, which assumes turbulent conditions (i.e., a high Reynolds number), can be formulated as

$$F_d = -\frac{1}{2}\rho A C_d \upsilon |\upsilon|$$

where $F_d =$ the drag force (N), $r =$ fluid density (kg/m^3), A = the frontal area of the object on a plane perpendicular to the direction of motion (m^2), υ = velocity (m/s), and $C_d =$ a dimensionless drag coefficient.

(a) Write the pair of differential equations for velocity and position (see Prob. 1.17) to describe the vertical motion of a sphere with diameter d (m) and a density of ρ_s (kg/m^3). The differential equation for velocity should be written as a function of the sphere's diameter.

(b) Use Euler's method with a step size of Δt = 2 s to compute the position and velocity of a sphere over the first 14 s. Employ the following parameters in your calculation: d = 125 cm, ρ = 1.3 kg/m^3, ρ_s = 2650 kg/m^3, and C_d = 0.475. Assume that the sphere has the initial conditions: $y(0) = -100$ m and $\upsilon(0) = -45$ m/s.

(c) Develop a plot of your results (i.e., y and υ versus t) and use it to graphically estimate when the sphere would hit the ground.

(d) Compute the value for the bulk second-order drag coefficient c_d' (kg/m). Note that, as described in Prob. 1.3, the bulk second-order drag coefficient is the term in the final differential equation for velocity that multiplies the term $\upsilon|\upsilon|$.

(a) The force balance can be written as

$$m\frac{dv}{dt} = mg - \frac{1}{2}\rho v|v| A C_d$$

Dividing by mass gives

$$\frac{dv}{dt} = g - \frac{\rho A C_d}{2m} v|v| \qquad (1)$$

The mass of the sphere is $\rho_s V$ where V = volume (m^3). The area and volume of a sphere are $\pi d^2/4$ and $\pi d^3/6$, respectively. Substituting these relationships gives

$$\frac{dv}{dt} = g - \frac{3\rho C_d}{4d\rho_s} v|v|$$

$$\frac{dx}{dt} = v$$

(b) The first step for Euler's method is

$$\frac{dv}{dt} = 9.81 - \frac{3(1.3)0.475}{4(1.25)2650}(-45)|-45| = 10.09312$$

$$\frac{dx}{dt} = -45$$

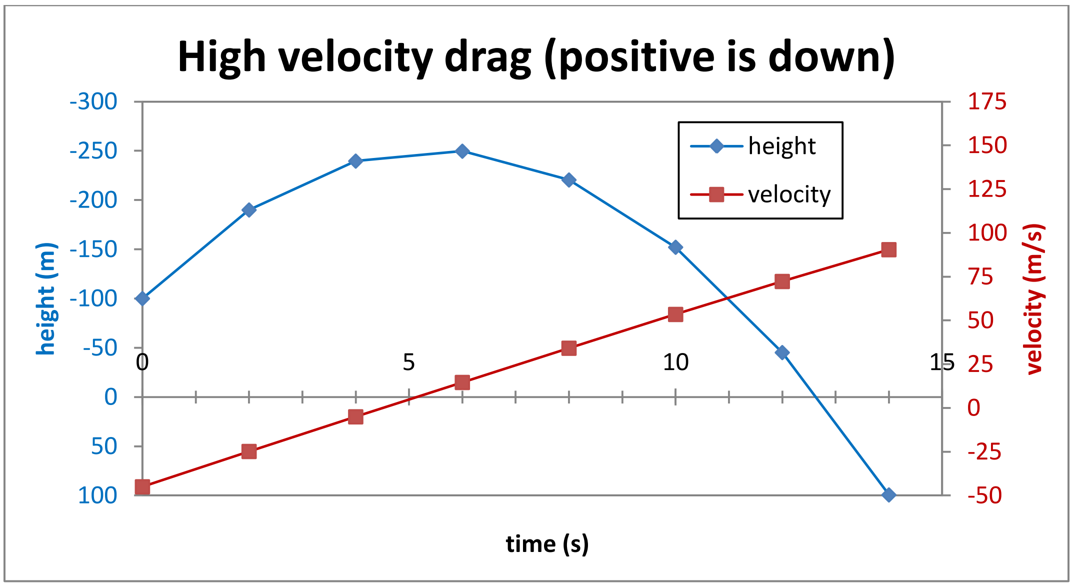
$$v = -45 + 10.09312(2) = -24.8138$$

$$\frac{dx}{dt} = -100 - 45(2) = -190$$

The remaining steps are shown in the following table:

t	*x*	*v*	*dx/dt*	*dv/dt*
0	-100	-45	-45	10.09312
2	-190	-24.8138	-24.8138	9.896085
4	-239.628	-5.02159	-5.02159	9.813526
6	-249.671	14.60546	14.60546	9.780176
8	-220.46	34.16581	34.16581	9.646798
10	-152.128	53.4594	53.4594	9.410432
12	-45.2094	72.28027	72.28027	9.079565
14	99.35116	90.4394	90.4394	8.666443

(c) The results can be graphed as (notice that we have reversed the left axis for the height, *x*, so that the positive elevations point down). The sphere will hit the ground around 12.6 seconds.



(d) Inspecting the differential equation for velocity (Eq. 1) indicates that the bulk drag coefficient is

$$c' = \frac{\rho A C_d}{2}$$

Therefore, for this case, because $A = \pi(1.2)^2/4 = 1.131$ m^2, the bulk drag coefficient is

$$c' = \frac{1.3(1.2272)0.475}{2} = 0.37889 \frac{\text{kg}}{\text{m}}$$

1.21 As depicted in Fig. P1.21, a spherical particle settling through a quiescent fluid is subject to three forces: the downward force of gravity (F_G), and the upward forces of buoyancy (F_B) and drag (F_D). Both the gravity and buoyancy forces can be computed with Newton's second law, with the latter equal to the weight of the displaced fluid. For laminar flow, the drag force can be computed with *Stokes's law*,

$$F_D = 3\pi\mu d\upsilon$$

where μ = the dynamic viscosity of the fluid (N s/m^2), d = the particle diameter (m), and υ = the particle's settling velocity (m/s). Note that the mass of the particle can be expressed as the product of the particle's volume and density ρ_s (kg/m^3) and the mass of the displaced fluid can be computed as the product of the particle's volume and the fluid's density ρ (kg/m^3). The volume of a sphere is $\pi d^3/6$. In addition, laminar flow corresponds to the case where the dimensionless Reynolds number, Re, is less than 1, where Re = $\rho d\upsilon/\mu$.

(a) Use a force balance for the particle to develop the differential equation for $d\upsilon/dt$ as a function of d, ρ, ρ_s, and μ.
(b) At steady-state, use this equation to solve for the particle's terminal velocity.
(c) Employ the result of **(b)** to compute the particle's terminal velocity in m/s for a spherical silt particle settling in water: d = 8 μm, ρ = 1 g/cm^3, ρ_s = 2.7 g/cm^3, and μ = 0.014 g/(cm s).
(d) Check whether flow is laminar.
(e) Use Euler's method to compute the velocity from $t = 0$ to 2^{-15} s with $\Delta t = 2^{-18}$ s given the provided parameters along with the initial condition: $\upsilon(0) = 0$.

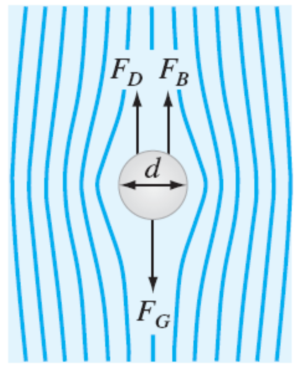


FIGURE P1.21

(a) A force balance on a sphere can be written as:

$$m\frac{dv}{dt} = F_{\text{gravity}} - F_{\text{buoyancy}} - F_{\text{drag}}$$

where

$$F_{\text{gravity}} = mg \qquad F_{\text{buoyancy}} = \rho V g \qquad F_{\text{drag}} = 3\pi\mu d v$$

Substituting the individual terms into the force balance yields

$$m\frac{dv}{dt} = mg - \rho V g - 3\pi\mu d v$$

Divide by m

$$\frac{dv}{dt} = g - \frac{\rho V g}{m} - \frac{3\pi\mu d v}{m}$$

Note that $m = \rho_s V$, so

$$\frac{dv}{dt} = g - \frac{\rho g}{\rho_s} - \frac{3\pi\mu d v}{\rho_s V}$$

The volume can be represented in terms of more fundamental quantities as $V = \pi d^3/6$. Substituting this relationship into the differential equation gives the final differential equation

$$\boxed{\frac{dv}{dt} = g\left(1-\frac{\rho}{\rho_s}\right) - \frac{18\mu}{\rho_s d^2}v}$$

(b) At steady-state, the equation is

$$0 = g\left(1-\frac{\rho}{\rho_s}\right) - \frac{18\mu}{\rho_s d^2}v$$

which can be solved for the terminal velocity

$$\boxed{v_\infty = \frac{g}{18}\frac{\rho_s - \rho}{\mu}d^2}$$

This equation is sometimes called *Stokes Settling Law*.

(c) Before computing the result, it is important to convert all the parameters into consistent units. For the present problem, the necessary conversions are

$$d = 8\ \mu\text{m} \times \frac{\text{m}}{10^6\ \mu\text{m}} = 8\times10^{-6}\,\text{m} \qquad \rho = 1\frac{\text{g}}{\text{cm}^3}\times\frac{10^6\ \text{cm}^3}{\text{m}^3}\times\frac{\text{g}}{10^3\ \text{kg}} = 1000\frac{\text{kg}}{\text{m}^3}$$

$$\rho_s = 2.7\frac{\text{g}}{\text{cm}^3}\times\frac{10^6\ \text{cm}^3}{\text{m}^3}\times\frac{\text{g}}{10^3\ \text{kg}} = 2700\frac{\text{kg}}{\text{m}^3} \qquad \mu = 0.014\frac{\text{g}}{\text{cm s}}\times\frac{100\ \text{cm}}{\text{m}}\times\frac{\text{kg}}{1000\ \text{g}} = 0.0014\frac{\text{kg}}{\text{m s}}$$

The terminal velocity can then computed as

$$v_\infty = \frac{9.81}{18}\frac{2700-1000}{0.0014}(8\times10^{-6})^2 = 4.23543\times10^{-5}\,\frac{\text{m}}{\text{s}}$$

(d) The Reynolds number can be computed as

$$\text{Re} = \frac{\rho d v}{\mu} = \frac{1000(8\times10^{-6})4.23543\times10^{-5}}{0.0014} = 0.000242024$$

This is far below 1, so the flow is very laminar.

(e) Before implementing Euler's method, the parameters can be substituted into the differential equation to give

$$\frac{dv}{dt} = 9.81\left(1-\frac{1000}{2700}\right) - \frac{18(0.0014)}{2700(0.000008)^2}v = 6.176667 - 145{,}833v$$

NOTE: calculations are shown to several digits, with actual calculations completed using double precision.

With $\Delta t = 2^{-15} = 3.8147\times10^{-6}\,\text{s}$, the first two steps for Euler's method are

$$v(3.8147\times10^{-6}) = 0 + (6.176667 - 145{,}833(0))\times3.8147\times10^{-6} = 2.3562\times10^{-5}$$

$$v(7.6294\times10^{-6}) = 2.5362\times10^{-5} + (6.176667 - 145{,}833(2.3562\times10^{-5}))\times3.8147\times10^{-6} = 3.4016\times10^{-5}$$

The remaining steps can be computed in a similar fashion as tabulated and plotted below:

t	*v*	*dv/dt*	*t*	*v*	*dv/dt*
0	0.0000E+00	6.176667	2.29×10^{-5}	4.2031E-05	0.047123
3.81×10^{-6}	2.3562E-05	2.740525	2.67×10^{-5}	4.2211E-05	0.020908
7.63×10^{-6}	3.4016E-05	1.215944	3.05×10^{-5}	4.2291E-05	0.009277
1.14×10^{-5}	3.8655E-05	0.539502	3.43×10^{-5}	4.2326E-05	0.004116
1.53×10^{-5}	4.0713E-05	0.239372	3.81×10^{-5}	4.2342E-05	0.001826
1.91×10^{-5}	4.1626E-05	0.106207			

1.22 As described in Prob. 1.21, in addition to the downward force of gravity (weight) and drag, an object falling through a fluid is also subject to a buoyancy force that is proportional to the displaced volume. For example, for a sphere with diameter d (m), the sphere's volume is $V = \pi d^3/6$ and its projected area is $A = \pi d^2/4$. The buoyancy force can then be computed as $F_B = -\rho V g$. We neglected buoyancy in our derivation of Eq. (1.9) because it is relatively small for an object like a parachutist moving through air. However, for a more dense fluid like water, it becomes more prominent.

(a) Derive a differential equation in the same fashion as Eq. (1.9), but include the buoyancy force and represent the drag force as described in Prob. 1.20.

(b) Rewrite the differential equation from **(a)** for the special case of a sphere.

(c) Use the equation developed in **(b)** to compute the terminal velocity (i.e., for the steady-state case). Use the following parameter values for a sphere falling through water: sphere diameter = 1.1 cm, sphere density = 2650 kg/m^3, water density = 1000 kg/m^3, and $C_d = 0.47$.

(d) Use Euler's method with a step size of $\Delta t = 0.03125$ s to numerically solve for the velocity from $t = 0$ to 0.25 s with an initial velocity of zero.

(a) A force balance on a sphere can be written as:

$$m\frac{dv}{dt} = mg - \rho V g - \frac{1}{2}\rho v|v| A C_d$$

(b) Dividing by mass gives

$$\frac{dv}{dt} = g - \frac{-\rho V g}{m} - \frac{\rho A C_d}{2m} v|v|$$

The mass of the sphere is $\rho_s V$ where V = volume (m^3). The area and volume of a sphere are $\pi d^2/4$ and $\pi d^3/6$, respectively. Substituting these relationships gives

$$\frac{dv}{dt} = g\left(1 - \frac{\rho}{\rho_s}\right) - \frac{3\rho C_d}{4\rho_s d} v|v|$$

(c) At steady state, for a sphere falling downward

$$0 = g\left(1 - \frac{\rho}{\rho_s}\right) - \frac{3\rho C_d}{4\rho_s d} v^2$$

which can be solved for

$$v = \sqrt{\frac{4 g \rho_s d}{3\rho C_d}\left(1 - \frac{\rho}{\rho_s}\right)}$$

Substituting the parameters gives

$$v = \sqrt{\frac{4(9.81)2650(0.011)}{3(1000)0.47}\left(1-\frac{1000}{2700}\right)} = 0.710711\frac{\text{m}}{\text{s}}$$

(d) Before implementing Euler's method, the parameters can be substituted into the differential equation to give

$$\frac{dv}{dt} = 9.81\left(1-\frac{1000}{2650}\right) - \frac{3(1000)0.47}{4(2650)(0.011)}v^2 = 6.1108113 - 12.09262v^2$$

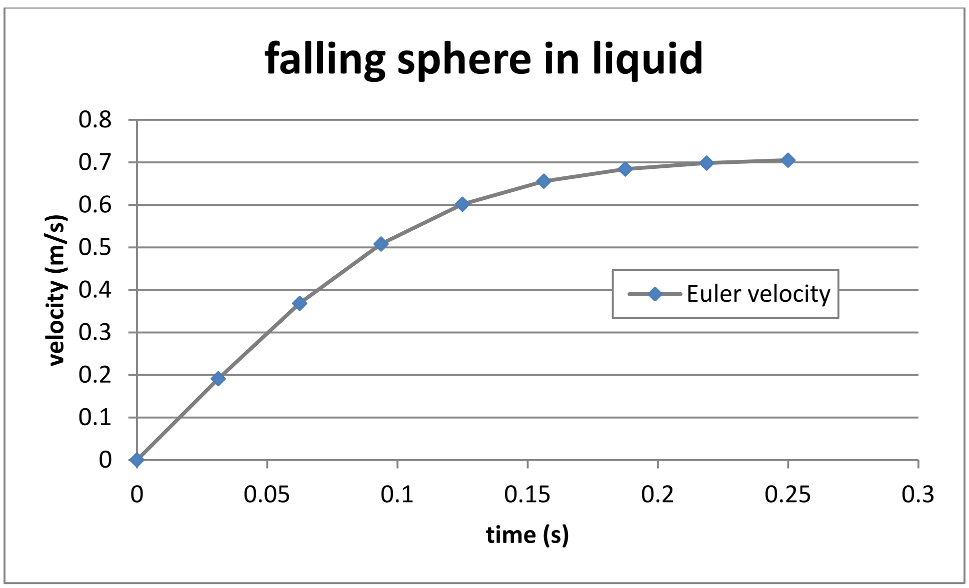
The first two steps for Euler's method are

$$v(0.03125) = 0 + (6.1108113 - 12.09262(0)^2)0.03125 = 0.190879$$

$$v(0.0625) = 0.193021 + (6.1108113 - 12.09262(0.190879)^2)0.03125 = 0.367989$$

The remaining steps can be computed in a similar fashion as tabulated and plotted below:

t	*v*	*dv*/*dt*	*t*	*v*	*dv*/*dt*
0	0	6.108113	0.15625	0.655475	0.912547
0.03125	0.190879	5.667523	0.1875	0.683992	0.450637
0.0625	0.367989	4.470583	0.21875	0.698075	0.21528
0.09375	0.507694	2.991196	0.25	0.704802	0.101152
0.125	0.601169	1.737785			



1.23 As depicted in Fig. P1.23, the downward deflection y (m) of a cantilever beam with a uniform load w (kg/m) can be computed as

$$y = \frac{w}{24EI}(x^4 - 4Lx^3 + 6L^2x^2)$$

where x = distance (m), E = the modulus of elasticity = 2×10^{11} Pa, I = moment of inertia = 3.3×10^{-4} m^4, w = 12,000 N/m, and L = length = 4 m. This equation can be differentiated to yield the slope of the downward deflection as a function of x:

$$\frac{dy}{dx} = \frac{w}{24EI}\left(4x^3 - 12Lx^2 + 12L^2x\right)$$

If y = 0 at x = 0, use this equation with Euler's method (Δx = 0.125 m) to compute the deflection from x = 0 to L. Develop a plot of your results along with the analytical solution computed with the first equation.

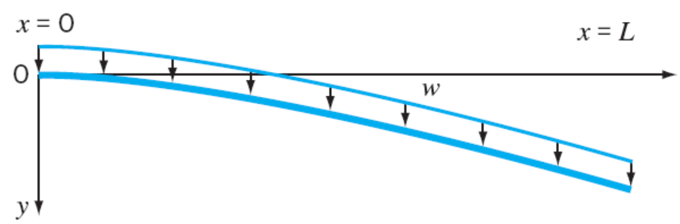


FIGURE P1.23
A cantilever beam.

Substituting the parameters into the differential equation gives

$$\frac{dy}{dx} = \frac{10000}{24(2\times10^{11})0.000325}\left(4x^3 - 12(4)x^2 + 12(4)^2 x\right)$$
$$= 2.5641\times10^{-5}\left(x^3 - 12x^2 + 48x\right)$$

The first step of Euler's method is

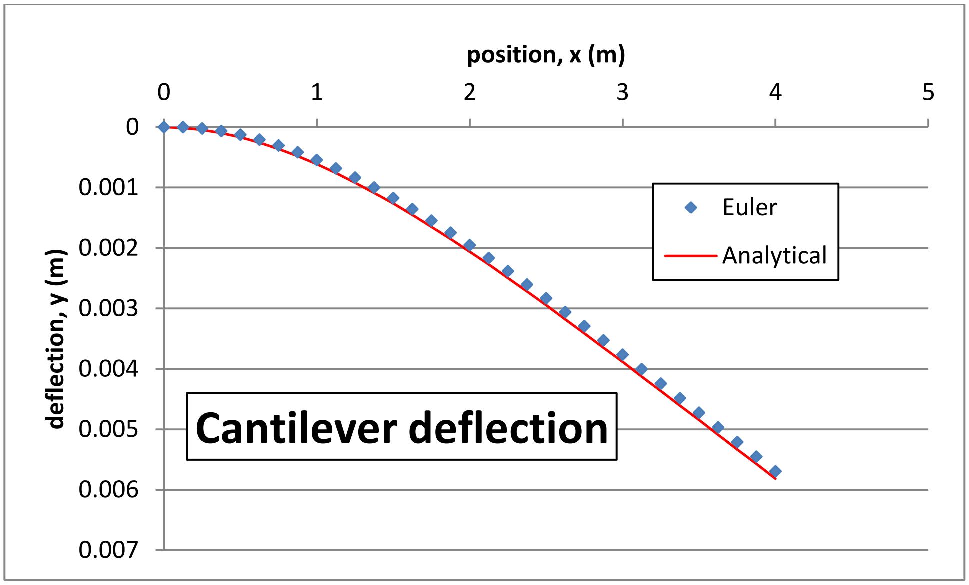
$$\frac{dy}{dx} = 2.5641\times10^{-5}\left((0)^3 - 12(0)^2 + 48(0)\right) = 0$$
$$y(0.125) = 0 + 0(0.125) = 0$$

The second step is

$$\frac{dy}{dx} = 2.5641\times10^{-5}\left((0.125)^3 - 12(0.125)^2 + 48(0.125)\right) = 0.000149$$
$$y(0.25) = 0 + 0.000149(0.125) = 1.86361\times10^{-5}$$

The remainder of the calculations along with the analytical solution are summarized in the following table and plot. Note that the results of the numerical and analytical solutions are close.

x	*y*-Euler	*dy/dx*	*y*-analytical	*x*	*y*-Euler	*dy/dx*	*y*-analytical
0	0	0	0	2.125	0.002165	0.00174	0.002275
0.125	0	0.000176	1.11E-05	2.25	0.002383	0.001777	0.002495
0.25	2.2E-05	0.000341	4.36E-05	2.375	0.002605	0.001809	0.002719
0.375	6.47E-05	0.000496	9.6E-05	2.5	0.002831	0.001837	0.002947
0.5	0.000127	0.00064	0.000167	2.625	0.003061	0.001861	0.003179
0.625	0.000207	0.000774	0.000256	2.75	0.003293	0.00188	0.003412
0.75	0.000304	0.000899	0.00036	2.875	0.003528	0.001896	0.003648
0.875	0.000416	0.001015	0.00048	3	0.003765	0.001909	0.003886
1	0.000543	0.001121	0.000614	3.125	0.004004	0.001919	0.004126
1.125	0.000683	0.001219	0.00076	3.25	0.004244	0.001927	0.004366
1.25	0.000835	0.001309	0.000918	3.375	0.004485	0.001932	0.004607
1.375	0.000999	0.001391	0.001087	3.5	0.004726	0.001936	0.004849
1.5	0.001173	0.001466	0.001266	3.625	0.004968	0.001938	0.005091
1.625	0.001356	0.001533	0.001453	3.75	0.00521	0.001939	0.005333
1.75	0.001548	0.001594	0.001649	3.875	0.005453	0.001939	0.005576
1.875	0.001747	0.001649	0.001851	4	0.005695	0.001939	0.005818
2	0.001953	0.001697	0.002061				



1.24 Use *Archimedes' principle* to develop a steady-state force balance for a spherical ball of ice floating in seawater (Fig. P1.24). The force balance should be expressed as a third-order polynomial (cubic) in terms of height of the cap above the water line (h), the seawater's density (ρ_f), the ball's density (ρ_s), and the ball's radius (r).

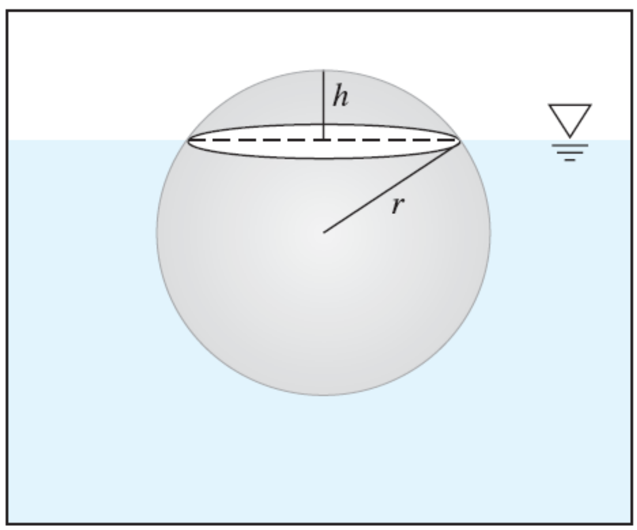


FIGURE P1.24

[Note that students can easily get the underlying equations for this problem off the web]. The volume of a sphere can be calculated as

$$V_s = \frac{4}{3}\pi r^3$$

The portion of the sphere above water (the "cap") can be computed as

$$V_a = \frac{\pi h^2}{3}(3r - h)$$

Therefore, the volume below water is

$$V_s = \frac{4}{3}\pi r^3 - \frac{\pi h^2}{3}(3r - h)$$

Thus, the steady-state force balance can be written as

$$\rho_s g \frac{4}{3}\pi r^3 - \rho_f g \left[\frac{4}{3}\pi r^3 - \frac{\pi h^2}{3}(3r - h)\right] = 0$$

Cancelling common terms gives

$$\rho_s \frac{4}{3} r^3 - \rho_f \left[\frac{4}{3} r^3 - \frac{h^2}{3}(3r - h)\right] = 0$$

Collecting terms yields

$$\frac{\rho_f}{3} h^3 - r\rho_f h^2 - (\rho_s - \rho_f)\frac{4}{3} r^3 = 0$$

1.25 Beyond fluids, *Archimedes' principle* has proven useful in geology when applied to solids on the earth's crust. Figure P1.25 depicts one such case where a lighter conical granite mountain "floats on" a denser basalt layer at the earth's surface. Note that the part of the cone below the surface is formally referred to as a *frustum*. Develop a steady-state force balance for this case in terms of the following parameters: basalt's density (ρ_b), granite's density (ρ_g), the cone's bottom radius (r), and the height above (h_1) and below (h_2) the earth's surface.

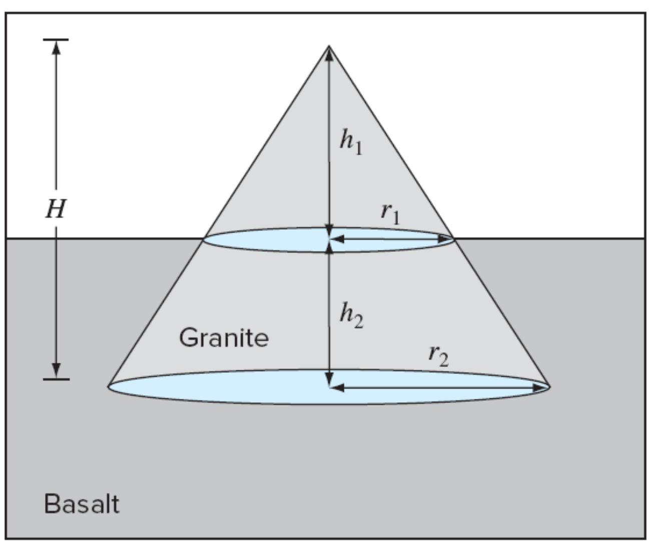


FIGURE P1.25

[Note that students can easily get the underlying equations for this problem off the web]. The total volume of a right circular cone can be calculated as

$$V_t = \frac{1}{3}\pi r_2^2 H$$

The volume of the frustum below the earth's surface can be computed as

$$V_b = \frac{\pi\left(H - h_1\right)}{3}\left(r_1^2 + r_2^2 + r_1 r_2\right)$$

Archimedes' principle says that, at steady state, the downward force of the whole cone must be balanced by the upward buoyancy force of the below ground frustum,

$$\frac{1}{3}\pi r_2^2 H g \rho_g = \frac{\pi\left(H - h_1\right)}{3}\left(r_1^2 + r_2^2 + r_1 r_2\right) g \rho_b \tag{1}$$

Before proceeding we have too many unknowns: r_1 and h_1. So before solving, we must eliminate r_1 by recognizing that using similar triangles ($r_1/h_1 = r_2/H$)

$$r_1 = \frac{r_2}{H} h_1$$

which can be substituted into Eq. (1) (and cancelling the g's)

$$\frac{1}{3}\pi r_2^2 H \rho_g = \frac{\pi\left(H - h_1\right)}{3}\left(\left(\frac{r_2}{H} h_1\right)^2 + r_2^2 + \frac{r_2^2}{H} h_1\right)\rho_b$$

Therefore, the equation now has only 1 unknown: h_1, and the steady-state force balance can be written as

$$\rho_s g \frac{4}{3}\pi r^3 - \rho_f g\left[\frac{4}{3}\pi r^3 - \frac{\pi h^2}{3}\left(3r - h\right)\right] = 0$$

Cancelling common terms gives

$$\rho_s \frac{4}{3} r^3 - \rho_f\left[\frac{4}{3} r^3 - \frac{h^2}{3}\left(3r - h\right)\right] = 0$$

and collecting terms yields

$$\frac{\rho_f}{3} h^3 - r\rho_f h^2 - \left(\rho_s - \rho_f\right)\frac{4}{3} r^3 = 0$$

1.26 Figure P1.26 shows the forces exerted on a hot air balloon system.

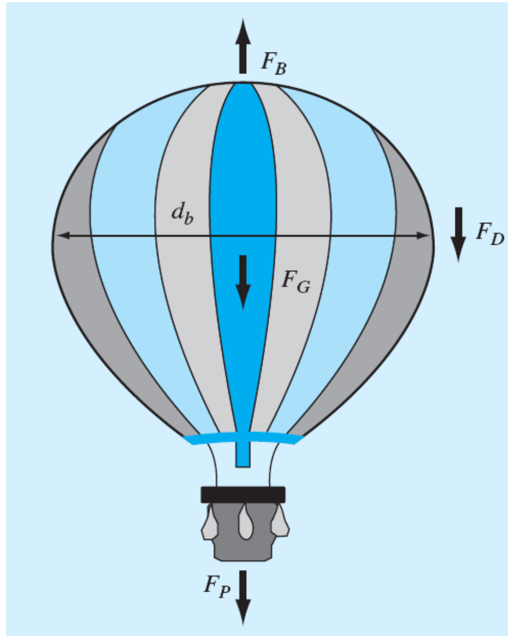


FIGURE P1.26

Forces on a hot air balloon: F_B = buoyancy, F_G = weight of gas, F_P = weight of payload (including the balloon envelope), and F_D = drag. Note that the direction of the drag is downward when the balloon is rising.

Formulate the drag force as

$$F_D = \frac{1}{2}\rho_a \upsilon^2 A C_d$$

where $r_a =$ air density (kg/m^3), υ = velocity (m/s), A = projected frontal area (m^2), and $C_d =$ the dimensionless drag coefficient ($\cong$ 0.47 for a sphere). Note also that the total mass of the balloon consists of two components:

$$m = m_G = m_P$$

where $m_G =$ the mass of the gas inside the expanded balloon (kg), and $m_P =$ the mass of the payload (basket, passengers, and the unexpanded balloon = 265 kg). Assume that the ideal gas law holds ($P = \rho RT$), that the balloon is a perfect sphere with a diameter of 17.3 m, and that the heated air inside the envelope is at roughly the same pressure as the outside air.

Other necessary parameters are:

Normal atmospheric pressure, P = 101,300 Pa

The gas constant for dry air, R = 287 J/(kg K)

The air inside the balloon is heated to an average temperature,

$$T = 100°\text{C}$$

The normal (ambient) air density, ρ = 1.2 kg/m^3

(a) Use a force balance to develop the differential equation for $d\upsilon/dt$ as a function of the model's fundamental parameters.

(b) At steady-state, calculate the particle's terminal velocity.

(c) Use Euler's method and Excel to compute the velocity from $t = 0$ to 60 s with $\Delta t = 2$ s given the previous parameters along with the initial condition: $\upsilon(0) = 0$. Develop a plot of your results.

(a) Use force balance to find *dv/dt*. Positive is defined as down, and the payload buoyancy, from the mass of air displaced due to the basket and passengers, is considered negligible.

$$F = F_G + F_P + F_D - F_B = F_{G+P} + F_D - F_B$$
$$m\frac{dv}{dt} = mg + \frac{1}{2}\rho_a v^2 AC_d - m_d g$$
$$\frac{dv}{dt} = g + \frac{1}{2m}\rho_a v^2 AC_d - \frac{m_d}{m}g$$

where m is total mass of the balloon gas and payload, $m = m_G + m_P$, and m_d is the mass of the displaced air

$$m_d = \rho_a V_b = \rho_a \frac{4\pi}{3}r^3 = \rho_a \frac{\pi}{6}d^3$$

and

$$m_G = \rho V_b = \frac{PV_b}{RT} = \frac{\pi P d^3}{6RT}$$

$$\frac{dv}{dt} = g + \frac{1}{2(m_G + m_P)}\rho_a v^2 AC_d - \frac{\pi\rho_a g d^3}{6(m_G + m_P)}$$
$$\frac{dv}{dt} = g - \frac{\pi\rho_a g d^3}{6(m_G + m_P)} + \frac{\rho_a AC_d}{2(m_G + m_P)}v^2$$

(b) find terminal velocity

$$0 = g + \frac{1}{2(m_G + m_P)}\rho_a v^2 AC_d - \frac{\pi\rho_a g d^3}{6(m_G + m_P)}$$
$$v^2 = \frac{2(m_G + m_P)}{\rho_a AC_d}\left(\frac{\pi\rho_a g d^3}{6(m_G + m_P)} - g\right) = \frac{\pi g d^3}{3\pi d^2 C_d} - \frac{2g(m_G + m_P)}{\pi\rho_a d^2 C_d}$$
$$v = \pm\sqrt{\frac{gd}{3C_d} - \frac{2g(m_G + m_P)}{\pi\rho_a d^2 C_d}}$$

given the parameters

$$m_G = \frac{\pi(101{,}300\text{ Pa})(17.3\text{ m})^3}{6(287\text{ Jkg}^{-1}\text{K}^{-1})(100 + 273.15\text{ °K})} = 2564.372\text{ kg}$$

$$v = -\sqrt{\frac{(9.81)(17.3)}{3(0.47)} - \frac{2(9.81)(2564.372 + 265)}{\pi(1.2)(17.3)^3(0.47)}} = -3.960155\text{ m/s}^2$$

part c) Simplify calculations by defining a constant and a coefficient

$$\frac{dv}{dt} = C_1 + C_2 v^2$$
$$C_1 = g - \frac{\pi\rho_a g d^3}{6(m_G + m_P)} \qquad C_2 = \frac{\pi\rho_a C_d d^2}{2(m_G + m_P)}$$

$$C_1 = 9.81 - \frac{p(1.2)(9.81)(17.3)^3}{6(2564.372 + 265)} = -1.46969$$

$$C_2 = \frac{p(1.2)(0.47)(17.3)^2}{2(2564.372 + 265)} = 0.093713$$

and the first two steps are

$$v(2) = v(0) + (C_1 + C_2 v(0)^2)\Delta t = 0 + (-1.46969 + 0.093713(0)^2)(2) = -2.93938$$

$$v(4) = -2.93938 + (-1.46969 + 0.093713(-2.93938)^2)(2) = -4.2594$$

and the rest of the calculations proceed similarly

t	*v*	*dv*/*dt*	*t*	*v*	*dv*/*dt*
0	0	-1.46969	32	-3.96017	9.29E-06
2	-2.93938	-0.66001	34	-3.96015	-4.5E-06
4	-4.2594	0.230504	36	-3.96016	2.18E-06
6	-3.79839	-0.11761	38	-3.96015	-1.1E-06
8	-4.03362	0.055035	40	-3.96016	5.12E-07
10	-3.92355	-0.02704	42	-3.96015	-2.5E-07
12	-3.97764	0.013005	44	-3.96015	1.2E-07
14	-3.95163	-0.00632	46	-3.96015	-5.8E-08
16	-3.96427	0.003058	48	-3.96015	2.82E-08
18	-3.95816	-0.00148	50	-3.96015	-1.4E-08
20	-3.96112	0.000718	52	-3.96015	6.62E-09
22	-3.95969	-0.00035	54	-3.96015	-3.2E-09
24	-3.96038	0.000169	56	-3.96015	1.55E-09
26	-3.96004	-8.2E-05	58	-3.96015	-7.5E-10
28	-3.96021	3.96E-05	60	-3.96015	3.64E-10
30	-3.96013	-1.9E-05