Chapter 3: Discrete Random Variables

3.1 The Notion of a Random Variable

3.1

Sample Space:
Carlos

	Michael \ Carlos	0	1	2
1/4	0	(0,0)	(0,1)	(0,2)
1/2	1	(1,0)	(1,1)	(1,2)
1/4	2	(2,0)	(2,1)	(2,2)
		1/4	1/2	1/4

Probabilities

	0	1	2
0	1/16	1/8	1/16
1	1/8	1/4	1/8
2	1/16	1/8	1/16

Mapping $S \to S_X$

	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

$$P[X=0] = P[(0,0)] = 1/16$$

$$P[X=1] = P[\{(1,0),(1,1),(0,1)\}] = \frac{1}{2}$$

$$P[X=2] = 3\times\frac{1}{16} + 2\times\frac{1}{8} = \frac{7}{16}.$$

3.2

(a) $S = \{1, 2, 3, 4, 5, 6\}$ $\quad p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = \frac{1}{6}$

where $p_j = P[\{j\}]$

(b)

S		S_X
1	$\longrightarrow$	0
2	$\longrightarrow$	1
3	$\nearrow$	
4	$\longrightarrow$	2
5	$\nearrow$	
6	$\longrightarrow$	3

(c)

$P[X=0] = p_1 = \frac{1}{6}$

$P[X=1] = p_2 + p_3 = \frac{2}{6}$

$P[X=2] = p_4 + p_5 = \frac{2}{6}$

$P[X=6] = p_6 = \frac{1}{6}$

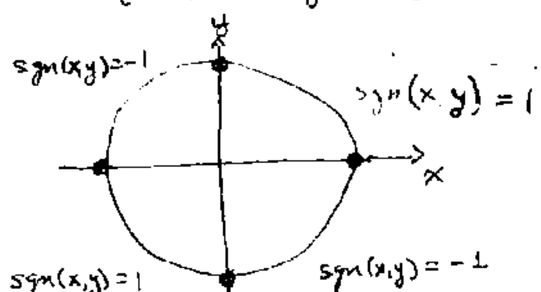
(d) $P[\cancel{X}Y=0] = p_1 + p_2 = \frac{2}{6}$ $\quad P[Y=1] = p_3 + p_4 = \frac{2}{6}$ $\quad P[Y=2] = p_5 + p_6 = \frac{2}{6}$

(e) $X=0$ corresponds to $\{1\}$

$Y=0$ corresponds to $\{1, 2\}$

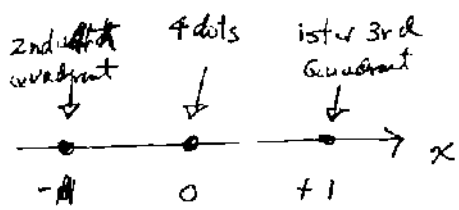
3.3

(a) $S = \{(x,y) : x^2 + y^2 = r^2\}$ where r = radius of circle



Outcomes from S occur uniformly along the circle.

sgn(xy) = 0 at the dots



(c) $P[X=-1] = P[\text{2nd + 3rd Quad}] = \frac{1}{2}$

$P[X=0] = P[\{(r,0), (0,r), (-r,0), (0,-r)\}]$

$= 0$

$P[X=1] = P[\text{1st + 3rd Quad}] = \frac{1}{2}$

3.4

a) $S = \{0000, 0001, \ldots, 1111\}$

$p_{0000} = p_{0001} = \ldots = p_{1111} = \frac{1}{16}$

b)

S	0000	0001	0010	...	1111
	↓	↓	↓		↓
S_X	0	1	2	...	15

c) $p_0 = p_1 = p_2 = \ldots = p_{15} = \frac{1}{16}$

d) $P[0b_1b_2b_3] = \frac{1}{4} \cdot \frac{1}{8} = \frac{1}{32}$ all $b_1b_2b_3$

$P[1b_1b_2b_3] = \frac{3}{4} \cdot \frac{1}{8} = \frac{3}{32}$ all $b_1b_2b_3$

$p_0' = p_1' = \ldots = p_7' = \frac{1}{32}$

$p_8' = p_9' = \ldots = p_{15}' = \frac{3}{32}$

3.5 Let A_i = Transmitter #1 sends a signal at time slot i

B_i = " #2 "

A signal gets through if $A_i B_i^c \cup A_i^c B_i$ occurs

Each experiment has 4 outcomes

(a) Experiment i

	A_i	A_i^c
B_i	1/4	1/4
B_i^c	1/4	1/4

Sample Space consists of a Cartesian product of the outcomes of the basic experiment

$S = (s_1, s_2, \ldots)$ where s_i is an outcome from basic experiment

(b) $X(s) = n$

if n is the first occurrence of $A_i B_i^c \cup A_i^c B_i$ in $s_1, s_2, \ldots$

(c) $P[A_i B_i^c \cup A_i^c B_i] = P[A_i B_i^c] + P[A_i^c B_i] = \frac{1}{2} = P[\text{success}]$

$P[X=k] = P[(k-1) \text{ failures}, 1 \text{ success}] = \left(\frac{1}{2}\right)^k$

3.6 $S = \{000, 111, 010, 101, 001, 110, 100, 011\}$

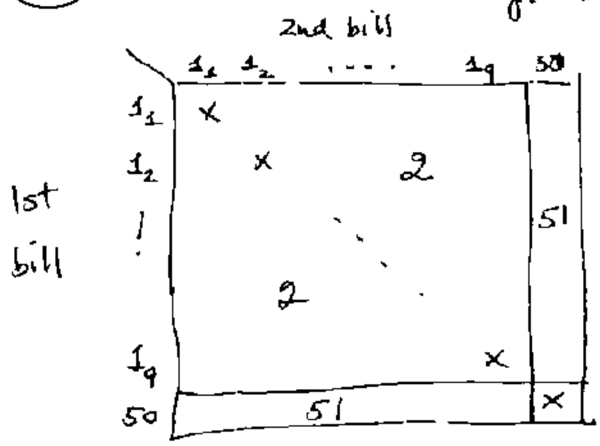
$X(s)$: 2, 2, 3, 3, 4, 4, 4, 4

$P[X=2] = P[\{000, 111\}] = \frac{1}{2}$

$P[X=3] = P[\{010, 101\}] = \frac{1}{4}$

$P[X=4] = P[\{001, 110, 100, 011\}] = \frac{1}{4}$

3.7 Draw 2 bills without replacement.



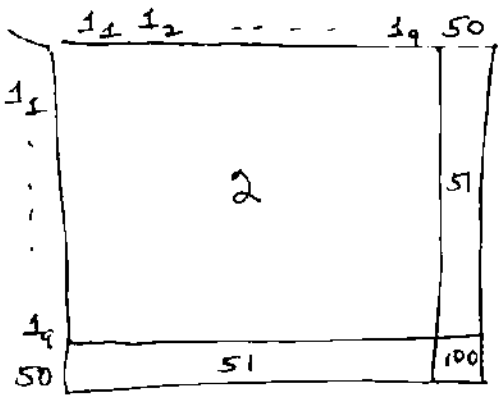
x not allowed

all other outcomes have probability $\frac{1}{9(10)} = \frac{1}{90}$

$$P[X=2] = \frac{9 \cdot 8}{90} = \frac{8}{10} = \frac{4}{5}$$

$$P[X=51] = \frac{9 \cdot 2}{90} = \frac{2}{10} = \frac{1}{5}$$

3.8 Draw 2 bills with replacement.



all outcomes have probability $\frac{1}{10(10)} = \frac{1}{100}$

$$P[X=2] = \frac{81}{100}$$

$$P[X=51] = \frac{18}{100}$$

$$P[X=100] = \frac{1}{100}$$

3.9) a) Let m be number of tails $0 \le m \le n$
then number of heads is $n-m$ and the difference is

$$Y = n-m-m = n-2m \qquad 0 \le m \le n$$

$$\therefore \; S_Y = \{-n, -n+2, \ldots, n-2, n\}$$

b) $P[Y=0] = P[n=2m] = P[m=\frac{n}{2}]$ for n even.

$P[Y=k] = P[n-2m=k] = P[m = \frac{n-k}{2}]$ for $n-k$ even

3.10
Let $S = \{b_1, b_2, \ldots, b_{2^m}\}$ be the sequence of m-bit passwords as covered by the hacker.
The target system picks a password at random from S.
$X(\xi)$ is the index of the selected password.

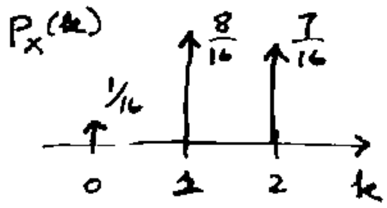
$S_X = \{1, 2, \ldots, 2^m\}$ where the value of X is selected at random from S_X.

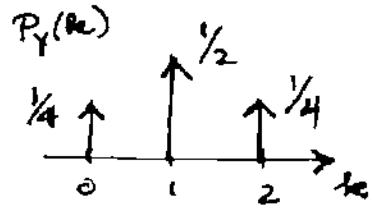
$P[i] = 2^{-m} \quad i \in S_X$.

3.2 Discrete Random Variables And Probability Mass Function

3.11

a)





the max function shifts probability mass to higher values of k

b) If Carlos uses a biased coin:

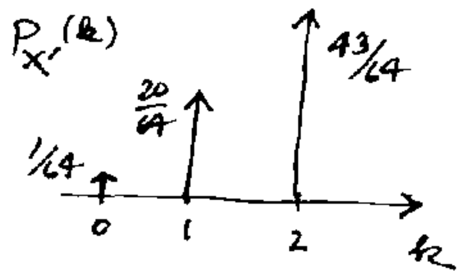
Michael \ Carlos	0	1	2	
0	0	1	2	1/4
1	1	1	2	1/2
2	2	2	2	1/4
	1/16	6/16	9/16	

$$P[X'=0] = \frac{1}{4}\frac{1}{16} = \frac{1}{64}$$

$$P[X'=1] = \frac{1}{16}\cdot\frac{1}{2} + \frac{6}{16}\cdot\frac{1}{2} + \frac{6}{16}\cdot\frac{1}{4}$$

$$= \frac{20}{64}$$

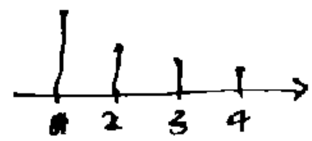
$$P[X'=2] = \frac{43}{64}$$



3.12

$1 = p_1 + p_2 + p_3 + p_4 = p_1\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) = \frac{25}{12} p_1 \quad p_1 = \frac{12}{25}$

$p_1 = \frac{12}{25} \quad p_2 = \frac{6}{25} \quad p_3 = \frac{4}{25} \quad p_4 = \frac{3}{25}$

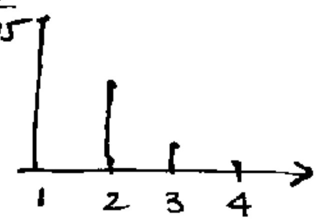


(b) $1 = p_1\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right) = \frac{15}{8} p_1$

$p_1 = \frac{8}{15} \quad p_2 = \frac{4}{15} \quad p_3 = \frac{2}{15} \quad p_4 = \frac{1}{15}$

(c) $1 = p_1\left(1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{64}\right) = \frac{105}{64} p_1$

$p_1 = \frac{64}{105} \quad p_2 = \frac{32}{105} \quad p_3 = \frac{8}{105} \quad p_4 = \frac{1}{105}$



pmf decays more steeply in each example

(d) $1 = p_1 \sum_{i=1}^{\infty} \frac{1}{i}$ does not converge so this pmf does not extend to $\{1, 2, \ldots\}$

$$1 = p_1 \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i = p_1 \frac{1}{1 - \frac{1}{2}} \Rightarrow p_1 = \frac{1}{2}.$$

this extends to the geometric pmf.

$$1 = p_1\left(1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^{1+2} + \left(\frac{1}{2}\right)^{1+2+3} + \ldots\right)$$

$$= p_1 \sum_{j=1}^{\infty} \left(\frac{1}{2}\right)^{j(j+1)/2}$$

this is a subseries of the geometric series so it converges

3.13 (a) $1 = \sum_{k=1}^{\infty} \frac{c}{k^2} = c \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ is a special case of the zeta function

$= 1.6449 \Rightarrow c = 0.608$

The sum of the first 100 terms gives $1.6349 \Rightarrow c \approx 0.611$

(b) $P[X > 4] = 1 - P[X \le 3] = 1 - \underbrace{c\left[1 + \frac{1}{4} + \frac{1}{9}\right]}_{0.8325}$

$= 0.1675$

(c) $P[6 \le X \le 8] = c\left[\frac{1}{36} + \frac{1}{49} + \frac{1}{64}\right] = 0.390$

3.14 $P[X \ge 8] = \sum_{k=8}^{15} p_k = \frac{8}{16} = \frac{1}{2}$

$P[Y \ge 8] = \sum_{k=8}^{15} p'_k = 8 \cdot \frac{3}{32} = \frac{24}{32} = \frac{3}{4}$

3.15

Terminal 2

Terminal 1		p	$1-p$
	$\frac{1}{2}$	$\frac{1}{2}p$	$\frac{1}{2}q$
	$\frac{1}{2}$	$\frac{1}{2}p$	$\frac{1}{2}q$

$P_{success} = \frac{1}{2}q + \frac{1}{2}p = \frac{1}{2}$ same

∴ The pmf of X is unchanged.

$$P[\text{Terminal 2 transmitted} \mid \text{success}] = \frac{P[\text{success and Terminal 2 transmitted}]}{P[\text{success}]}$$

$$= \frac{\frac{1}{2}p}{\frac{1}{2}} = p$$

This suggests that terminal 2 should always transmit (at the expense of terminal 1).

3.16 from problem 3.7b:

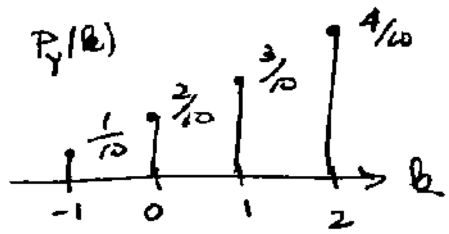
a) $P[X > 2] = 1 - P[X = 2] = \frac{1}{5}$.

$P[X > 50] = P[X = 51]$
$= \frac{1}{5}$

b) $P[X > 2] = 1 - P[X = 2] = \frac{19}{100}$

$P[X > 50] = P[X = 51] + P[X = 100]$
$= \frac{19}{100}$

3.17
a)
$Y = 0 + 2 = 2$ with prob. $4/10$
$Y = -1 + 2 = 1$ " $3/10$
$Y = -2 + 2 = 0$ " $2/10$
$Y = -3 + 2 = -1$ " $1/10$



b) $P[Y = 2] = 4/10$

c) $P[Y > 0] = P[Y = 2] + P[Y = 1] = \frac{4}{10} + \frac{3}{10} = \frac{7}{10}$

3.18 Let X be number of transmissions until ~~start~~ success.

$$P[X \le k] = \sum_{j=1}^{k} \left(\frac{1}{2}\right)^j = \frac{1}{2} \sum_{j=0}^{k-1} \left(\frac{1}{2}\right)^{j} = \frac{1}{2} \, \frac{1 - \left(\frac{1}{2}\right)^k}{\frac{1}{2}} = 1 - \left(\frac{1}{2}\right)^k$$

$1 - \left(\frac{1}{2}\right)^k = 0.99 \qquad \left(\frac{1}{2}\right)^k = 0.01$

$k = \frac{\ln 100}{\ln 2} = 6.64 \approx 7$

start sending refresh messages
7 × 10 seconds before expiry time