Physics 410/510 HW#1

3.1.1, 3.1.2, 3.2.3, 3.2.20, 3.2.34, 3.2.36

$\boxed{3.1.1}$ (a) $\underset{\sim}{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$$|\underset{\sim}{A}| = 1 \cdot \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 1 \cdot (-1) = -1 \;//$$

(b) $\underset{\sim}{A} = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 1 & 2 \\ 0 & 3 & 1 \end{pmatrix}$

$$|\underset{\sim}{A}| = 1 \cdot \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} + 2 \cdot (-1) \cdot \begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix}$$

$$= -5 + (-2) \cdot 3 = -11 \;//$$

(c) $\underset{\sim}{A} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$

$$|\underset{\sim}{A}| = \frac{1}{\sqrt{2}} \cdot \left\{ \sqrt{3} \cdot (-1)^{3+4} \cdot \begin{vmatrix} 0 & \sqrt{3} & 0 \\ \sqrt{3} & 0 & 0 \\ 0 & 2 & \sqrt{3} \end{vmatrix} \right\}$$

$$= \frac{\sqrt{3}}{\sqrt{2}} \cdot (-1) \cdot \sqrt{3} \cdot (-1)^{2+1} \cdot \begin{vmatrix} \sqrt{3} & 0 \\ 2 & \sqrt{3} \end{vmatrix}$$

$$= \frac{3}{\sqrt{2}} \cdot 3 = 9/\sqrt{2} \;//$$

$\boxed{3.1.2}$ $\begin{pmatrix} 1 & 3 & 3 \\ 1 & -1 & 1 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, or $\underset{\sim}{A}|x\rangle = |\emptyset\rangle$

Test : $|\underset{\sim}{A}| = 1 \cdot (-1)^3 \cdot \begin{vmatrix} 3 & 3 \\ 1 & 3 \end{vmatrix} + (-1) \cdot (-1)^4 \cdot \begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} + 1 \cdot (-1)^5 \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}$

$$= (-1) \cdot 6 + (-1) \cdot (-3) + (-1) \cdot (-5) = 2.$$

$\therefore$ since $|\underset{\sim}{A}| \neq 0$; only solution is trivial : $x = y = z = 0$.

$\boxed{3.2.3}$ Let $\vec{r} = c_1 \vec{r}_1 + c_2 \vec{r}_2$.

By definition, $\underset{\sim}{A}\,\vec{r} = \sum_j a_{ij} r_j$

$$= \sum_j a_{ij}\left(c_1 r_{1j} + c_2 r_{2j}\right)$$

$$= \sum_j \left(c_1 a_{ij} r_{1j} + c_2 a_{ij} r_{2j}\right)$$ since c_i, a_{ij}, r_j are all numbers

$$= c_1 \sum_j a_{ij} r_{1j} + c_2 \sum_j a_{ij} r_{2j}$$

$$= c_1 \underset{\sim}{A}\,\vec{r}_1 + c_2 \underset{\sim}{A}\,\vec{r}_2$$

$\boxed{3.2.20}$ $\underset{\sim}{L}^{+} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\underset{\sim}{L}^{-} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

(The vectors $|-1\rangle$, $|0\rangle$, $|+1\rangle$ represent the 3 spin projection states J_z of a spin-1 system — $2j+1 = 3$ states.)

$\underset{\sim}{L}^{+}|-1\rangle = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = |0\rangle$. (raise by $\Delta m = 1$)

$\underset{\sim}{L}^{-}|-1\rangle = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \vec{\phi}$. (lower below m_{min})

$\underset{\sim}{L}^{+}|0\rangle = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = |1\rangle$. (raise by $\Delta m = 1$)

$\underset{\sim}{L}^{-}|0\rangle = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = |-1\rangle$. (lower by $\Delta m = -1$)

$\underset{\sim}{L}^{+}|1\rangle = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \vec{\phi}$. (raise above m_{max}.)

$\underset{\sim}{L}^{-}|1\rangle = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = |0\rangle$. (lower by $\Delta m = -1$)

where $\vec{\phi} \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

$\boxed{3.2.34}$ Find $\underset{\sim}{M}_L \ni \underset{\sim}{M}_L \underset{\sim}{A}$ gives $\underset{\sim}{A}$ but with:

(a) $\text{row}_i \to K\,\text{row}_i$, K constant

Let $\underset{\sim}{C} = \underset{\sim}{M}_L \underset{\sim}{A}$, $C_{nj} = \sum_{\ell} m_{n\ell}\, a_{\ell j}$

where $C_{nj} = a_{nj}$, except $C_{ij} = K a_{ij}$, $j = 1, \ldots, n$

Note that in $\sum_{\ell} m_{n\ell}\, a_{\ell j}$, the $\ell = i^{\text{th}}$ row of $\underset{\sim}{A}$ is only multiplied by the i^{th} column of $\underset{\sim}{M}_L$.

$\therefore$ we want $\underset{\sim}{M}_L = \underset{\sim}{1}$ except the i^{th} diagonal element is K:

$$\underset{\sim}{M}_L = \begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \ddots & 0 & & \\ \vdots & & & K & & \\ & & & 0 & 1 & \ddots \\ 0 & \cdots & & \vdots & & 1 \end{pmatrix}$$

or

$$m_{n\ell} = \delta_{n\ell} \cdot \left(1 + (K-1)\,\delta_{i\ell}\right) \; //$$

Verify:

$$C_{nj} = \sum_{\ell} \left(\delta_{n\ell}\, a_{\ell j} + \delta_{n\ell}\,\delta_{i\ell}\,(K-1)\, a_{\ell j} \right)$$

$$= a_{nj} + (K-1)\,\delta_{in}\, a_{nj}$$

$$= \begin{cases} a_{nj}, & n \neq i \\ K a_{nj}, & n = i \end{cases} \quad \checkmark$$

(b) $\text{row}_i \to \text{row}_i - K \cdot \text{row}_m$

or $$C_{nj} = \begin{cases} a_{nj}, & n \neq i \\ a_{nj} - K a_{mj}, & n = i, \; j = 1, 2, \ldots, n \end{cases}$$

Similar to previous case

$$m_{n\ell} = \delta_{n\ell} \cdot \left(1 - \left(\frac{K a_{mj}}{a_{\ell j}}\right) \delta_{i\ell}\right) . //$$

Verify: $$C_{nj} = \sum_{\ell} \left[\delta_{n\ell}\, a_{\ell j} - a_{\ell j}\, \delta_{n\ell}\, \delta_{i\ell} \left(\frac{K a_{mj}}{a_{\ell j}} \right) \right]$$

$$= a_{nj} - \delta_{in} \left(\frac{K a_{mj}}{a_{nj}} \right) a_{nj}$$

$$= \begin{cases} a_{nj}, & n \neq i \\ a_{nj} - K a_{mj}, & n = i \end{cases} \quad \checkmark$$

(c) $\text{row}_i \longleftrightarrow \text{row}_m \qquad (a_{ij} \leftrightarrow a_{mj} \;, j = 1, 2, \cdots, n)$

Use the Kronecker δ again

$$C_{nj} = \sum_{\ell} m_{n\ell}\, a_{\ell j}$$

Try $\quad m_{n\ell} = \delta_{n\ell} \left\{ 1 + \delta_{i\ell} \left(\frac{a_{mj}}{a_{\ell j}} \right) + \delta_{m\ell} \left(\frac{a_{ij}}{a_{\ell j}} \right) \right\}$ //

$$C_{nj} = \sum_{\ell} a_{\ell j} \left[\delta_{n\ell} + \delta_{n\ell}\delta_{i\ell} \left(\frac{a_{mj}}{a_{\ell j}} \right) + \delta_{n\ell}\delta_{m\ell} \left(\frac{a_{ij}}{a_{\ell j}} \right) \right]$$

$$= \begin{cases} a_{nj} \;, & n \neq i \;, n \neq m \\ a_{ij} \cdot \left(\frac{a_{mj}}{a_{ij}} \right) = a_{mj} \;, & n = i \\ a_{mj} \cdot \left(\frac{a_{ij}}{a_{mj}} \right) = a_{ij} \;, & n = m \end{cases}$$

$\boxed{3.2.36}$ $\quad \underset{\sim}{A} = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 4 \end{pmatrix}$ $\quad$ Use Gauss - Jordan elim. to find $\underset{\sim}{A}^{-1}$.

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 4 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\Downarrow$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 1 \\ 1 & 1 & 4 \end{pmatrix} \quad \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{row}_1 \to \text{row}_1 - \text{row}_2$$

$\Downarrow$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 1 \\ 0 & 0 & 7/2 \end{pmatrix} \quad \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -1/2 & 1 \end{pmatrix} \qquad \text{row}_3 \to \text{row}_3 - \tfrac{1}{2}\text{row}_2$$

$\Downarrow$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7/2 \end{pmatrix} \quad \begin{pmatrix} 1 & -1 & 0 \\ -2 & 22/7 & -2/7 \\ 0 & -1/2 & 1 \end{pmatrix} \qquad \begin{aligned} &\text{row}_2 \to \text{row}_2 - 2\,\text{row}_1 \text{, and} \\ &\text{row}_2 \to \text{row}_2 - \tfrac{2}{7}\text{row}_3 \end{aligned}$$

$\Downarrow$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & -1 & 0 \\ -1 & 11/7 & -1/7 \\ 0 & -1/7 & 2/7 \end{pmatrix} \qquad \begin{aligned} &\text{row}_2 \to \tfrac{1}{2}\text{row}_2 \\ &\text{row}_3 \to \tfrac{2}{7}\text{row}_3 \end{aligned}$$

Check: $\frac{1}{7}\begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 4 \end{pmatrix}\begin{pmatrix} 7 & -7 & 0 \\ -7 & 11 & -1 \\ 0 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; $\quad \therefore \underset{\sim}{A}^{-1} = \frac{1}{7}\begin{pmatrix} 7 & -7 & 0 \\ -7 & 11 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ //